Straight and Narrow Problem

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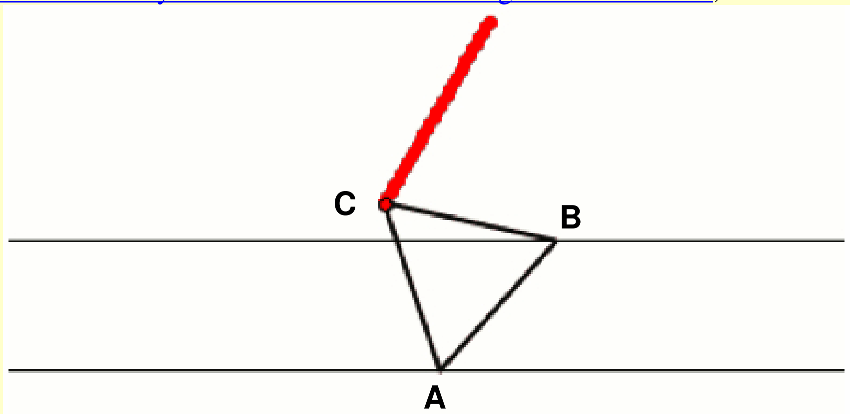
The following interesting behavior was found at the Futility Closet website:

Straight and Narrow (11 January 2016)

(http://www.futilitycloset.com/2016/01/11/straight-and-narrow-4/, retrieved 2/5/16)



A pleasing fact from David Wells' *Archimedes Mathematics Education Newsletter*:

Draw two parallel lines. Fix a point **A** on one line and move a second point **B** along the other line. If an equilateral triangle is constructed with these two points as two of its vertices, then as the second point moves, the third vertex **C** of the triangle will trace out a straight line.

Thanks to reader Matthew Scroggs for the tip and the GIF.

This is rather amazing and cries out for a proof. It also raises the question of how anyone noticed this behavior in the first place. I proved the result with calculus (see below), but I wonder if there is a slicker way that makes it more obvious.

My Solution

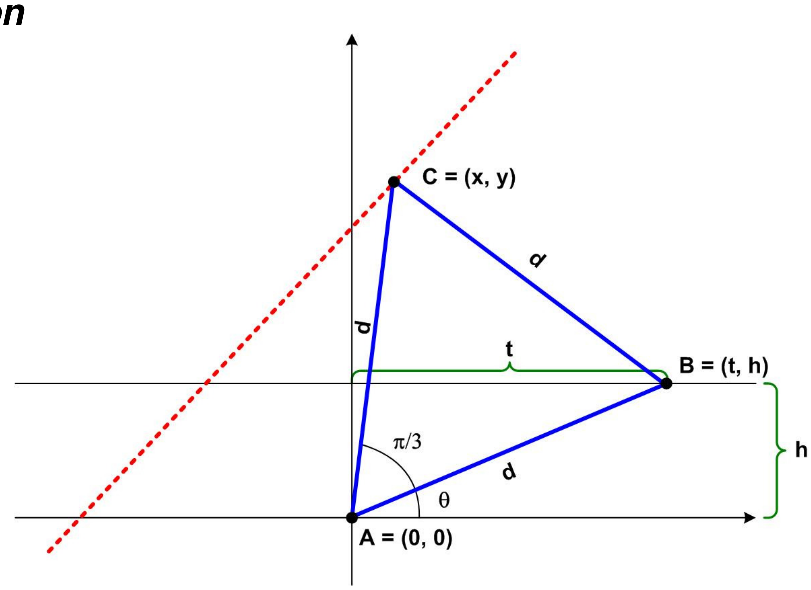


Figure 1 Straight and Narrow Parameterization

We parameterize the original figure as shown in Figure 1. What we are interested in is the red dashed curve (locus of points (x, y)) generated by the moving point **C**. We assume the parallel lines are separated by a distance h, and the point **B** along the top parallel line is a distance t from the vertical y-axis. The motion of **B**, then, is captured by the change in t. The distance between the points **A**, **B**, and **C** is given by d. As an intermediate parameterization, it will be helpful to consider the angle θ that the line **AB** makes with the horizontal x-axis. Since the triangle is equilateral, the interior angle at vertex **A** is 60° or $\pi/3$ radians.

Thus we can express x and y in terms of t via the equations:

$$\begin{aligned} x &= d \cos(\theta + \pi/3) \\ y &= d \sin(\theta + \pi/3) \end{aligned} \tag{1}$$

where

$$\begin{aligned} t &= d \cos\theta \\ h &= d \sin\theta \end{aligned} \tag{2}$$

Using sum of angles trig identities for equations (1) and substituting the values from equations (2) yields

$$\begin{aligned} x(t) &= \tfrac{1}{2}(t - h\sqrt{3}) \\ y(t) &= \tfrac{1}{2}(h + t\sqrt{3}) \end{aligned} \tag{3}$$

Now the instantaneous slope of the curve generated by **C** is given by the derivative dy/dx. But since x and y are given parametrically as functions of t, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}} = \sqrt{3}\,,$$

which is a constant. This means the curve is a **straight line** with constant slope $\sqrt{3}$.

Notice that $\sqrt{3}$ is the tangent of 60° or $\pi/3$ radians. So the line intersects the parallel lines at a 60° angle. This can be seen in the original animation when the point **C** crosses the bottom parallel line and the line **CB** lines up with the generated red line.