Regiomontanus 1471 Problem

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This is an old problem I had seen before. Here is Dave Wells's rendition ([1] p.202):

Johannes Müller, named Regiomontanus after the Latin translation of Körnigsberg, his city of birth, later made famous by Euler, proposed this problem in 1471. … it is usually put in this form …: **From what distance will a statue on a plinth appear largest to the eye [of a mouse!]?** If we approach too close, the statue appears foreshortened, but from a distance it is simply small.

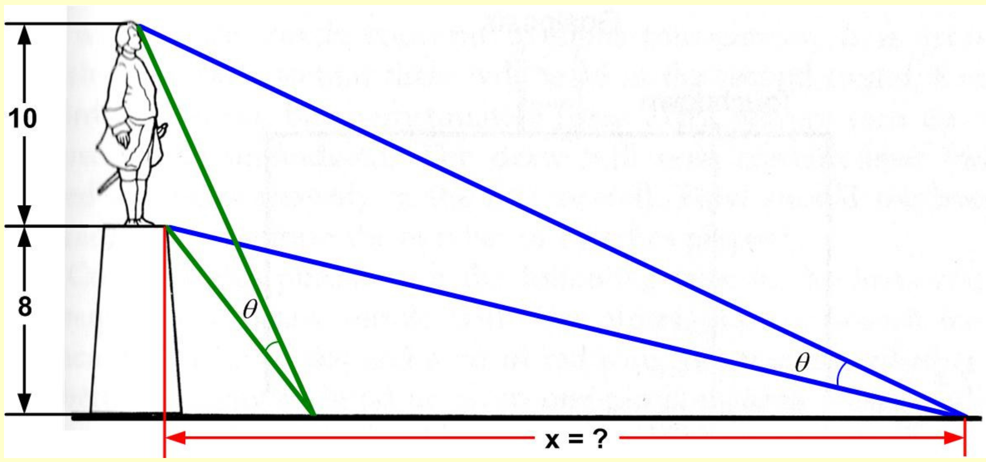

Figure 1 Maximizing View of Statue

I have added height numbers in feet for concreteness (as well as the mouse qualification, since the angles are measured from ground level). So the problem is to find the distance x such that the angle θ is maximal.

Solution – Geometry

Wells offers a geometric solution that I naturally did not think of, but I vaguely recall that this is the traditional solution. Wells does not compute actual values, so I thought I would add those steps, along with a detailed justification for the geometric solution.

The key is to superimpose the problem on angles inscribed in circles. Again, the "knowledge representation" or mathematical model is the heart of the matter and usually takes the most imagination. Figure 2 shows the viewing angles θ inscribed in a circle. Since they are both inscribed in the same circle in this example and subtend the same arc of the circle, their values are the same, which, recall, are one half the value of the central angle subtending the same arc.

Figure 3 shows what happens as we shrink the circles intersecting the base line. First, notice that the centers of all circles passing through the end points of the vertical viewing interval must lie on the perpendicular bisectors of the viewing interval. Second, central angles increase as the circles shrink, because consider the right triangle formed by one-half the central angle and the vertical leg of one-half the viewing interval. The angles labeled in the figure satisfy

$$\alpha + \beta = 90°$$

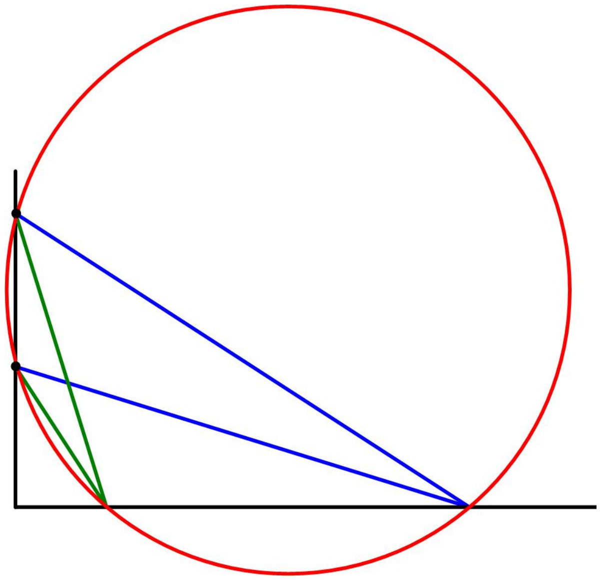

Figure 2 Inscribed Angles in a Circle

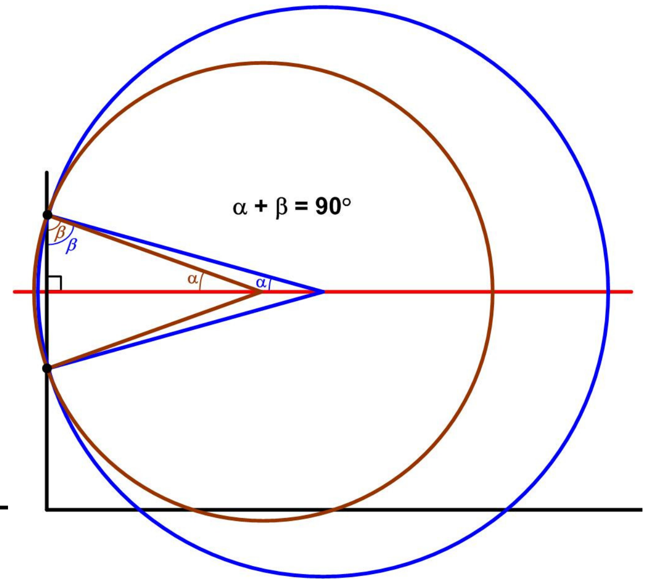

Figure 3 Central Angle Increases as Circle Decreases

Therefore, as the circle shrinks, the angle β decreases, which means the central angle α must increase. And conversely, when the circle increases, the angle β increases, and the central angle α decreases.

So the idea is to pick the smallest circle that intersects the baseline, namely, a circle tangent to the baseline at one point (Figure 4).

Once we have the circle giving us the largest viewing angle θ, then it is a straight-forward exercise to compute the distance x. We see that the radius of the circle must be the sum of the plinth height plus one-half the height of the statue, or $8 + 5 = 13$ feet. Then from the Pythagorean theorem we get the distance x to be

$$\sqrt{(13^2 - 5^2)} = \sqrt{12^2} = 12 \text{ feet.}$$

Now for the pedestrian calculus solution.

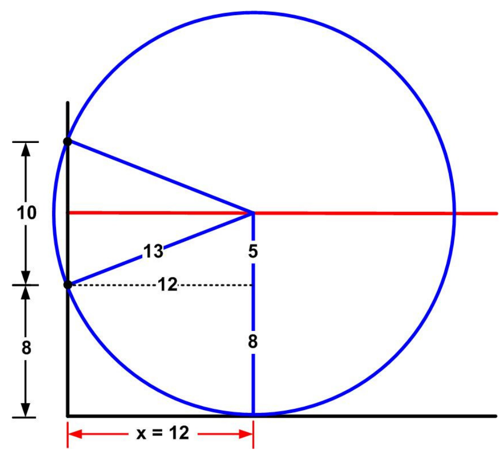

Figure 4 Solution

Solution – Calculus

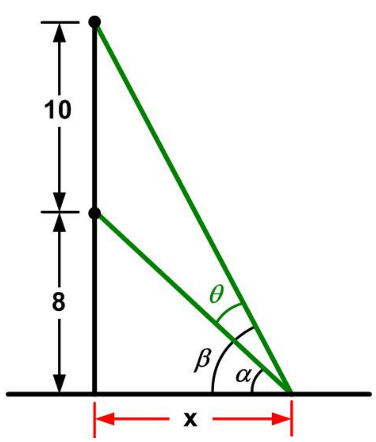

Figure 5 Calculus Approach

Figure 5 shows the setup for the problem. We have

$$\tan\theta = \tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha}$$

or

$$\tan\theta = \frac{10x}{x^2 + 8\cdot 18}$$

Maximizing $\tan\theta$ with respect to x for $0 < \theta < \pi/2$ is equivalent to maximizing θ in the interval. Taking derivatives, since $8\cdot 18 = 16\cdot 9$, we have

$$\sec^2\theta\frac{d\theta}{dx} = 10\frac{16\cdot 9 - x^2}{(x^2 + 16\cdot 9)^2}$$

This expression vanishes when $x^2 = 16{\cdot}9$ or $x = 4{\cdot}3 = 12$. And clearly since $\theta \to 0$ when x gets either very small or very large, it must be a maximum value for θ when the derivative vanishes. Thus x = 12 is where θ is maximum, just as we saw in the geometric solution.

References

[1] Wells, David, *You Are a Mathematician: A Wise and Witty Introduction to the Joy of Numbers*, John Wiley & Sons, New York, 1995