Cascading Squares Problem

(2 October 2018)

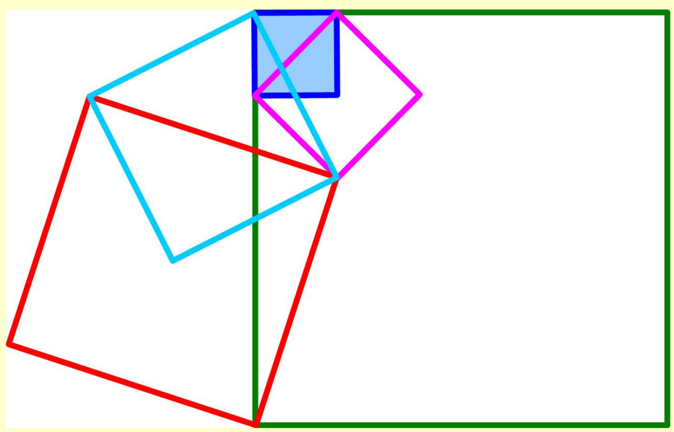
Jim Stevenson

Here is another imaginative geometry problem from Catriona Shearer's twitter account;

Catriona Shearer @Cshearer41 10:17 AM - 1 Oct 2018

(https://twitter.com/Cshearer41/status/1046811295834607618, retrieved 10/1/2018)

What fraction of the largest square is shaded?



Solution

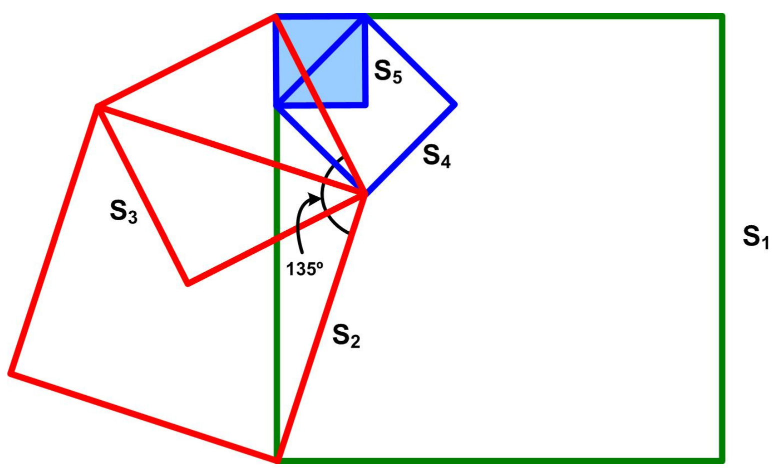


Figure 1 Annotated Squares

Label the edges the squares in descending size as S_1, S_2, S_3, S_4, S_5 as shown in Figure 1. Notice that the red set of squares is similar to the blue set, so that the green line between vertices in the red set (S_1) is to the red line in the blue set (S_3) as side S_3 is to side S_5. That is,

$$S_1/S_3 = S_3/S_5 \Rightarrow S_5/S_1 = S_3^2/S_1^2 \Rightarrow S_5^2/S_1^2 = (S_3^2/S_1^2)^2 \qquad (1)$$

Furthermore we have

$$S_2^2 = 2\ S_3^2 \text{ and so } S_2 = \sqrt{2}\ S_3. \qquad (2)$$

By the Law of Cosines we have (using equation (2))

$$S_1^2 = S_2^2 + S_3^2 - 2\,S_2\,S_3 \cos 135° = 2\,S_3^2 + S_3^2 - 2\,\sqrt{2}\,S_3^2\,(-1/\sqrt{2}) = 5\,S_3^2 \qquad (3)$$

Therefore, from equation (1) and equation (3) we have

$$S_5^2/S_1^2 = (S_3^2/S_1^2)^2 = (1/5)^2$$

So the ratio of the area of the smallest (shaded) square to the largest square is 1/25. That is, the largest square is 25 times the size of the smallest.