Star Sum of Angles Problem

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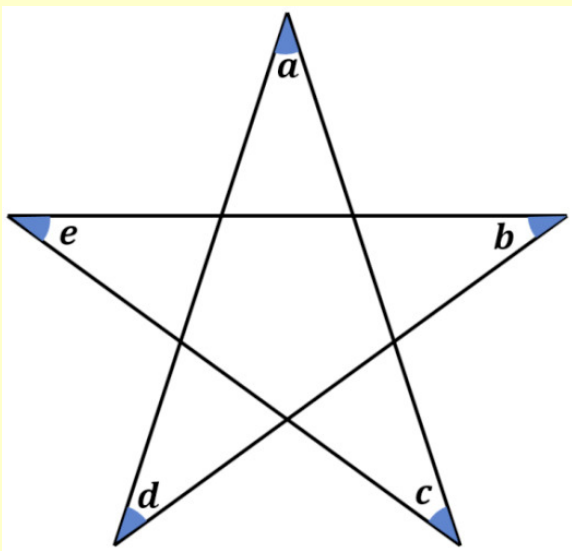
This problem posted by Presh Talwalkar offers a variety of solutions, but I didn't quite see my favorite approach for such problems. So I thought I would add it to the mix.

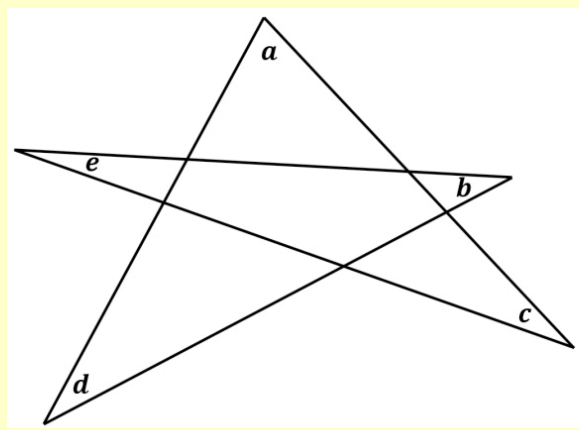
(https://mindyourdecisions.com/blog/2018/12/03/what-is-the-sum-of-angles-in-a-star-challenge-from-india/, retrieved 12/6/2018)

What Is The Sum Of Angles In A Star? Challenge From India

Posted December 3, 2018 By Presh Talwalkar.

Thanks to Nikhil Patro from India for suggesting this! What is the sum of the corner angles in a regular 5-sided star? What is $a + b + c + d + e = ?$





Here's a bonus problem: if the star is not regular, what is $a + b + c + d + e = ?$

Talwalkar Answer To Sum Of Angles In A Star

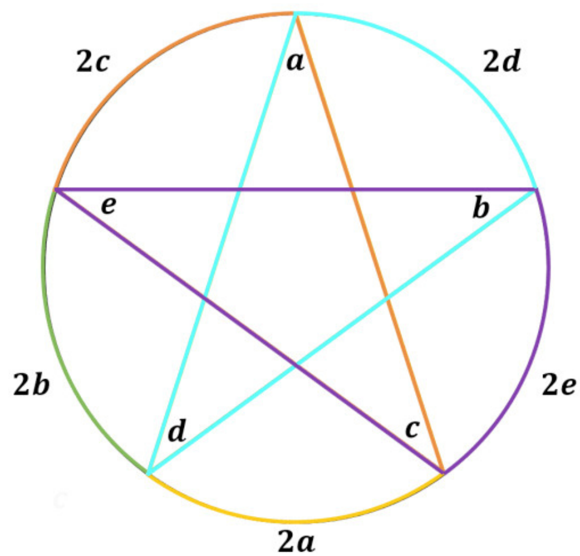
In the video, I explain an intuitive way to see the answer is 180 degrees. If you place a pen along one of the sides, and then rotate it through the 5 angles, you will end up with the pen in the same spot but flipped 180 degrees. You can see an animation of that here: star pentagon angle sum animation.[1]

While that is not a proof, it suggests an answer of 180 degrees, which might give you the idea to think about half of a circle or the sum of angles in a triangle. These concepts can be used to prove the result.

[1] https://puzzling.stackexchange.com/a/17703 JOS: This is solution #1 at the site: https://puzzling.stackexchange.com/questions/17681/five-angles-in-a-star. Solution #2 is essentially a duplicate of #1.

Proof For A Regular Star Pentagon

There is a wonderful proof for a regular star pentagon. A regular star pentagon is symmetric about its center so it can be inscribed in a circle. From there, we use the fact that an inscribed angle has a measure that is half of the arc it subtends. So we get a figure like this:



Since a circle measures 360 degrees, and the arcs combine to be the entire circle, we get:

$$2a + 2b + 2c + 2d + 2e = 360 \text{ degrees}$$

Now we divide this equation by 2 and magically we have the answer!

$$a + b + c + d + e = 180 \text{ degrees}$$

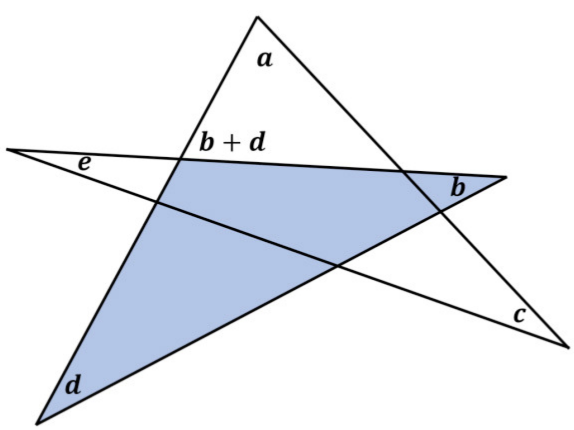
Now you could extend this proof for the more general case of an irregular star pentagon. You could do this by showing the sum of the corner angles is unchanged as you move a corner (proof here[2]). Thus you can always re-arrange an irregular star pentagon into a regular one, and since the total angle sum is unchanged, the irregular one must also have a measure of 180 degrees.

But there's another proof that is more direct that I prefer.

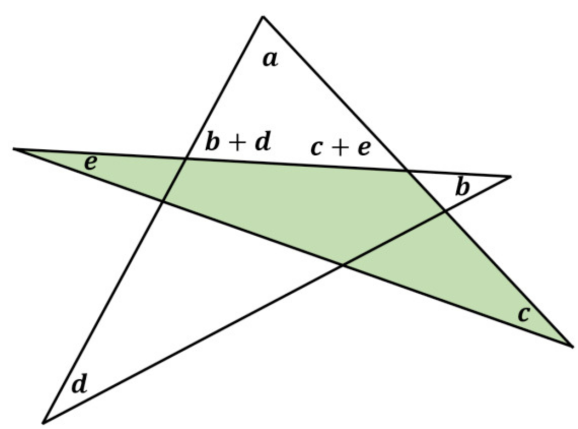
Proof For Any Star Pentagon

The key is to consider triangles.

The exterior angle to the triangle with corners b and d has an angle measure $b + d$:

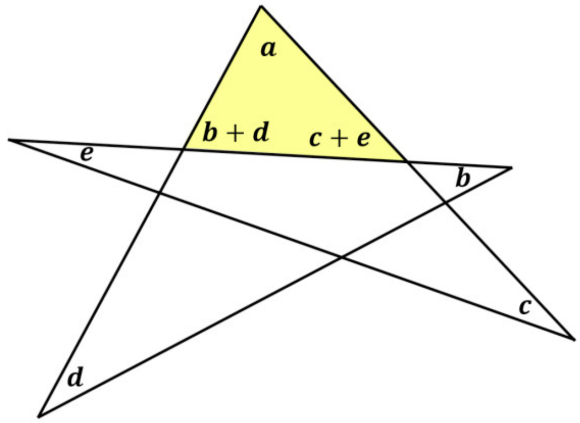


Similarly, the exterior angle to the triangle with corners c and e has an angle measure $c + e$:



[2] https://puzzling.stackexchange.com/a/17688/37015 JOS: This is solution #5 at the site: https://puzzling.stackexchange.com/questions/17681/five-angles-in-a-star.

Thus the top triangle has angles a, $b + d$, and $c + e$:



Since the sum of the angles in a triangle is 180 degrees, and this triangle has the sum of all the corner angles, we are done!

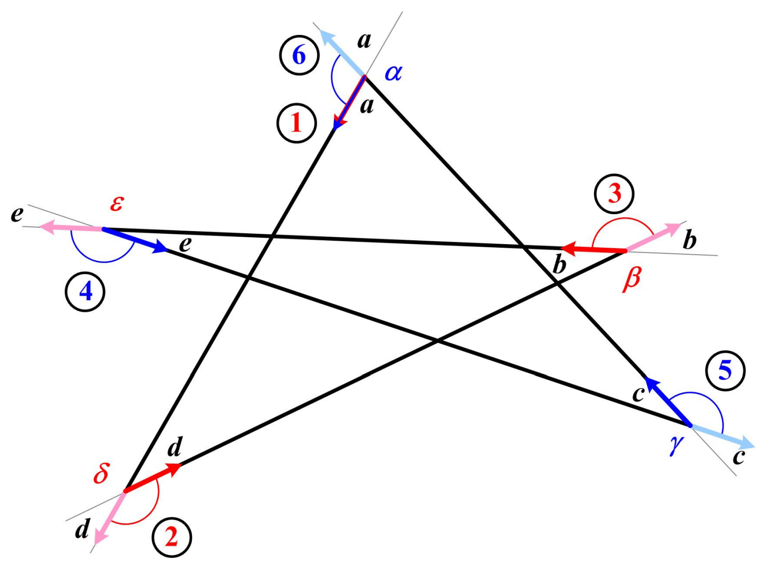
$a + b + c + d + e = 180$ degrees

There are many other ways to prove the result too! And you can then investigate other star polygons and closed curves—see the “further reading” link.

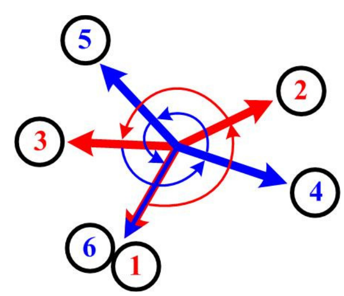
Further reading: http://www.math.nsysu.edu.tw/~wong/papers/soa-SEAM-formatted.pdf

My Proof For Any Star Pentagon

My approach employs a technique I use for measuring the interior angles of any closed, multi-lateral figure. In fact, Talwalkar seemed to allude to the idea with his rotating pen in the beginning of his discussion above. Only instead of considering the angles inside the star, we look at the angles swept out by each rotation of the pen, or arrow as you slide it along the sides of the star as in the figure to the right.



Beginning with the red arrow at the vertex with angle a (1), we slide it down the side of the star to the vertex with angle d and rotate it until it lies along the adjacent side of the star (2). We continue sliding and rotating at each vertex of the star until we return to the initial vertex with the arrow lying on top of its original position (6). Then the arrow has rotated through an accumulated sum of angles $\delta + \beta + \varepsilon + \gamma + \alpha$. Since the arrow returns to its original position, it must have rotated an integral number of 360°. In fact, the diagram at right shows the arrow made two complete rotations, that is, 720°. (The red arrows represent the first rotation and the blue arrows the second.) So



$$\delta + \beta + \varepsilon + \gamma + \alpha = 720° = 4 \times 180°.$$

On the other hand, the angles swept out by the arrow are the supplements of the vertex angles that we are interested in. That is,

$$\alpha + a = 180°$$
$$\beta + b = 180°$$

$$\gamma + c = 180°$$
$$\delta + d = 180°$$
$$\varepsilon + e = 180°$$

So when we add these five equations, we have

$$4 \times 180° + (a + b + c + d + e) = 5 \times 180° \Rightarrow a + b + c + d + e = 180°$$

which shows the sum of the vertex angles is the same for any star. Clearly this technique will work on similar figures with more sides.

(I also use this method to find the interior angles of a regular polygon, since I can never remember the formula. For example, for a regular octagon, the sliding-rotating arrow will sweep out 8 equal angles and make one complete rotation of 360°. Therefore, the exterior angle it sweeps out at each vertex is 360°/8 = 45°, which means 180° – 45° = 135° is the interior angle at each vertex.)

I checked the approaches given in Talwalkar's reference to https://puzzling.stackexchange.com/questions/17681/five-angles-in-a-star and his reference in "further reading". Neither employed my method exactly, though solution #7 in the stackexchange came the closest with its reference to the Gauss-Bonnet Theorem. Also my approach is a bit reminiscent of winding numbers, which are valuable tools for finding critical points, zeros of complex polynomials, or whether a point is inside or outside a closed curve.