Square Wheels

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I came across the following problem from an Italian high school exam on the British *Aperiodical* website presented by Adam Atkinson:[1]

There have been various stories in the Italian press and discussion on a Physics teaching mailing list I'm accidentally on about a question in the maths exam for science high schools in Italy last week. The paper appears to be online.[2] *(Ed. – Here's a copy of the first part of this four-part question, reproduced for the purposes of criticism and comment)*

PROBLEMA 1

Si può pedalare agevolmente su una bicicletta a ruote quadrate? A New York, al MoMath-Museum of Mathematics si può fare, in uno dei padiglioni dedicati al divertimento matematico (figura 1). È però necessario che il profilo della pedana su cui il lato della ruota può scorrere soddisfi alcuni requisiti.

In figura 2 è riportata una rappresentazione della situazione nel piano cartesiano Oxy: il quadrato di lato $DE = 2$ (in opportune unità di misura) e di centro C rappresenta la ruota della bicicletta, il grafico della funzione $f(x)$ rappresenta il profilo della pedana.

Figura 1

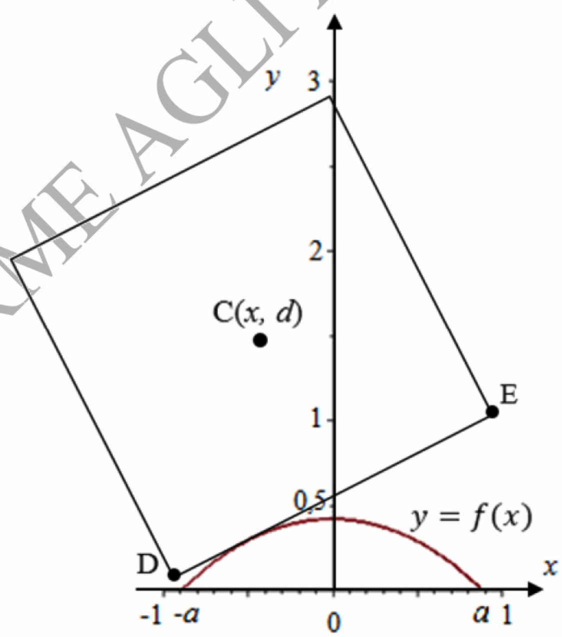

Figura 2

1) Sulla base delle informazioni ricavabili dal grafico in figura 2, mostra, con le opportune argomentazioni, che la funzione:

$$f(x) = \sqrt{2} - \frac{e^x + e^{-x}}{2} \qquad x \in \mathbb{R}$$

rappresenta adeguatamente il profilo della pedana per $x \in [-a\,;\,a]$; determina inoltre il valore degli estremi a e $-a$ dell'intervallo.

[1] http://aperiodical.com/2017/06/square-wheels-in-an-italian-maths-exam/, retrieved 7/5/2017

[2] http://www.istruzione.it/esame_di_stato/201617/Licei/Ordinaria/I043_ORD17.pdf

The question asks students to confirm that a given formula is the shape of the surface needed for a comfortable ride on a bike with square wheels. (Asking what the formula was with no hints would clearly have been harder.) It then asks what shape of polygon would work on another given surface.

What do people think? Would this be a surprising question at A-level in the UK or in the final year of high school in the US or elsewhere? Of course, I don't know how similar this question might be to anything in the syllabus in licei scientifici. …

My Solution

I had seen videos of riding a square-wheeled bicycle over a corrugated surface before, but I had never inquired about the nature of the surface. So I thought it would be a good time to see if I could prove the surface (cross-section) shown would do the job.

The first thing I noticed is that the equation for the curve given is actually $f(x) = \sqrt{2} - \cosh(x)$, that is, the curve represents an inverted catenary of some type where the hyperbolic cosine is given by $\cosh(x) = (e^{x} + e^{-x})/2$. It is not necessary to express things in terms of hyperbolic functions, but it streamlines the presentation.

I have redrawn the Italian exam's Figura 2 with some edits as my Figure 1. Notice we have $f(-a) = f(a) = 0$, which means $\cosh(-a) = \cosh(a) = \sqrt{2}$. I assumed the corner of the wheel touches the bottom of the dip between the repeated curves, namely the x-axis. Since the square is 2x2, that means the square is standing on its $2\sqrt{2}$ diagonal at the points (-a,0) and (a,0) with the center directly

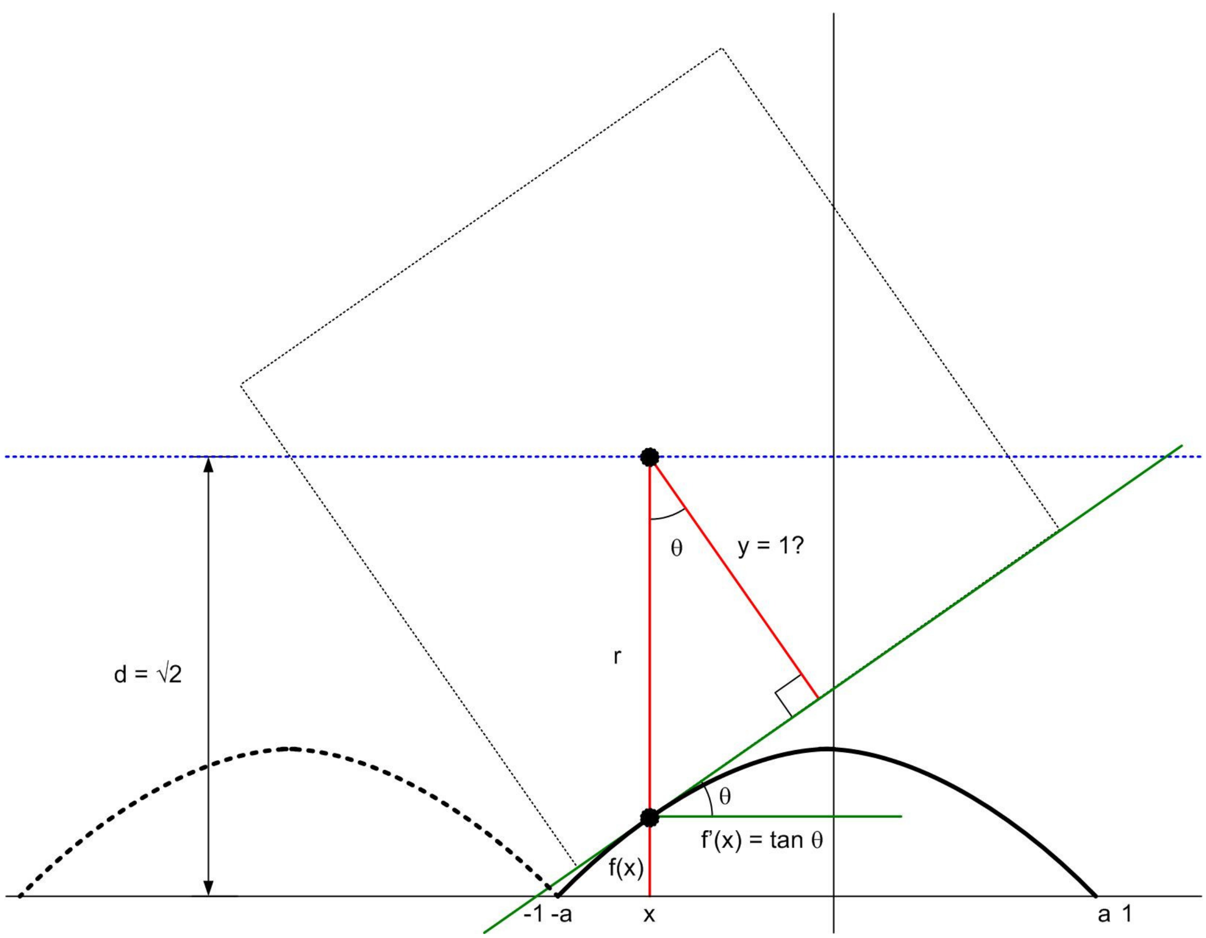

Figure 1 Square Wheel Solution

overhead these points at a distance of one-half the diagonal. To be touching (tangent to) the left end of the curve segment at the point (-a,0) means the angle the edge of the square (tangent line) makes with the horizontal (x-axis) is 45° or π/4 radians, that is, its slope must be 1.

At this point we transition to calculus to show the desired properties. The slope of the tangent line to the curve f(x) (tan θ in Figure 1) is given by the derivative f'(x) at the point of tangency. It is easy to see that

$$f'(x) = -(e^{x} - e^{-x})/2 = -\sinh(x)$$

The following, easily derived, property of hyperbolic functions will streamline our discussion:

$$\cosh(x)^2 - \sinh(x)^2 = 1$$

Now to check that when the corner of the square reaches the points (-a, 0) or (a, 0), the slope of the tangent to the curve is 1 or –1 respectively. $f'(a) = -\sinh(a)$ and $f'(-a) = -\sinh(-a) = \sinh(a)$. But $\sinh(a)^2 = \cosh(a)^2 - 1 = 2 - 1 = 1$ (recall $\cosh(-a) = \cosh(a) = \sqrt{2}$). Therefore $a > 0 \Rightarrow \sinh(a) > 0 \Rightarrow \sinh(a) = 1 \Rightarrow f'(-a) = 1$. Similarly, $-a < 0 \Rightarrow \sinh(-a) = -\sinh(a) < 0 \Rightarrow f'(a) = -\sinh(a) = -1$. So the property holds.

Now to the solution. Consider the point (x, d) in Figure 1. The vertical line from (x, d) down to the x-axis at x intersects the curve at (x, f(x)). The slanted green line represents the tangent line of the curve at that point. Consider the perpendicular from the point (x, d) down to the tangent line. We will designate its length y. If for all x in the interval [-a, a], y has constant length 1, then we effectively have a 2x2 square centered at (x, d) and lying on the tangent line, that is, touching the curve. So a 2x2 square wheel rotating along the curve has its axle at (x, d), that is, at a constant distance $d = \sqrt{2}$ from the x-axis.

From Figure 1 we have

$$r = \sqrt{2} - f(x) = \cosh(x)$$

and

$$y = r\cos\theta = \cosh(x)\cos\theta$$

$$= \frac{\cosh(x)}{\sqrt{\sec^2\theta}} \qquad [\cos\theta \ge 0 \text{ for } -\pi/2 \le \theta \le \pi/2]$$

$$= \frac{\cosh(x)}{\sqrt{1+\tan^2\theta}}$$

$$= \frac{\cosh(x)}{\sqrt{1+\sinh(x)^2}} \qquad [f'(x) = \tan\theta = -\sinh(x)]$$

$$= \frac{\cosh(x)}{\sqrt{\cosh(x)^2}} = 1 \qquad \text{Q.E.D.}$$

Discussion

This is not a trivial problem. It took me quite a while to work it out (two days! but I messed up some of the hyperbolic expressions—sigh). I certainly think it would be quite a challenge for a senior in high school. I also think it would take a little familiarity with hyperbolic functions to come up with the approach. And it certainly needs calculus.

I finally checked the full original Italian test sheet[3] after doing the above analysis. Atkinson only included part 1) of the question. It turns out there are 3 more parts which actually seem to be hints about how to proceed, ending in part 4) with an suggestion that looks a lot like my approach. (I don't read Italian, so it is difficult to be sure what each part is doing.) That makes the problem more accessible for a high school student. But I have to confess to enjoying tackling the question blind.

One other interesting feature is that the tangent point of the square on the curve is always directly below the center of the square. This falls out of the argument and was not evident initially. For example, if a circle is rotating around another circle, the tangent point is on the line joining the centers of the two circles, and that does not lie directly below the center of the rotating circle except at the beginning when it is directly over the lower circle. When I began the problem, I was working with the entire square and trying to figure out where the tangent point was. This caused a lot of confusion. Once I switched to the perpendicular on the tangent line (ignoring any relation to a square), things became clearer.

It seems to me that by choosing appropriate values of a along the catenary curve f(x), other regular polygon shapes for the wheels could be easily accommodated (the distance d would have to be one-half the longest diameter of the polygon). These points would be where the angle of the tangent line with respect to the x-axis equals π/n,[4] where the polygon has n sides.

References

I did a Google search and came up with some more references, but they also did not provide a solution, except for Wagon's, which didn't start with the solution but rather derived it:

1. Stan Wagon, "The Ultimate Flat Tire", Math Horizons, February **1999**, 14-17. (https://www.maa.org/sites/default/files/pdf/upload_library/22/Evans/february_1999_14.pdf, retrieved 7/6/2017).

I haven't fully digested Wagon's paper yet. It is a bit hard to follow. He parameterizes the "wheel" via a radius function of the angle of rotation. I am not yet sure what "radius" he means, so I have some study to do.

2. Kuczmarski, Fred, "Roads and Wheels, Roulettes and Pedals," *Mathematics Monthly* 118, The Mathematical Association Of America, June–July **2011**, pp.479-496. (https://www.semanticscholar.org/paper/Roads-and-Wheels-Roulettes-and-Pedals-Kuczmarski/678dff781f05cf39c4444ee746266e29b6f97e19, retrieved 7/6/2017)

Kuczmarski's paper discusses different shaped wheels and corrugated surfaces.

3. Propp, James, "The Lessons of a Square-Wheeled Trike," *Mathematical Enchantments*, 15 July **2015**. (https://mathenchant.wordpress.com/2015/07/15/the-lessons-of-a-square-wheeled-trike/, retrieved 7/6/2017)

Propp has images and videos from the Museum of Math (MoMath) that demonstrate the tricycle shown in the Italian exam's Figura 1 above.

These references also include references of their own.

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[3] http://www.istruzione.it/esame_di_stato/201617/Licei/Ordinaria/I043_ORD17.pdf

[4] This angle is found by subtracting one-half of an interior angle of the regular polygon from 90°, that is, $\pi/2 - ½(\pi - 2\pi/n) = \pi/n$