Exponential Yarn

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Reading Tanya Khovanova's recent blog post reminded me of the earliest math problem my father posed to me at the dinner table one night. I grew up in an age when the family all ate together at the dinner table. We often addressed the major controversy of the day, played games, or considered puzzles. This night my father posed if fruit flies doubled their number each minute and you put two fruit flies in a quart jar that was filled in an hour, when was it half filled?

Here is Tanya Khovanova's posting:

(https://blog.tanyakhovanova.com/2018/11/the-annoyance-of-hyperbolic-surfaces/, retrieved 12/1/2018)

The Annoyance of Hyperbolic Surfaces

Tanya Khovanova, 27 November 2018, 05:19 pm

I do not like making objects with my hands. But I lived in Soviet Russia. So I knew how to crochet, knit, and sew — because in Russia at that time, we didn't have a choice. I was always bad at it. The only thing I was good at was darning socks: I had to do it too often. By the way, I failed to find a video on how to darn socks the same way my mom taught me.

Then I came to the US. I suddenly found myself in a rich society, where it was cheaper to buy new stuff than to spend the time doing things with my hands. So I happily dropped my craftsmanship.

After not working with my hands for 28 years, one day I needed hyperbolic surfaces for my classes and I couldn't find any to buy. Hyperbolic surfaces are famous for providing an example when the Euclid's Fifth axiom doesn't work. These hyperbolic surfaces look flat locally, so you can continue a line in any given direction. If you draw a line on such a surface and pick a point that is not on the line, then you can draw many lines through the point that are parallel to the given line.

My students are more important than my dislike of crochet, so I decided to just do it myself. I asked my friend Debbie, who knows how to crochet, for advice, and she gave me more than advice. She gave me a hook and a piece of yarn and reminded me how to work the hook. She started me with a small circle. After that all I had to do was add two stitches for each stitch on the perimeter of the circle.[1] The finished product is this green ballish thing that looks like a brain, as in the photo.

Outside the starting circle, each small surface segment of this "brain" looks the same, making the "brain" a **surface of constant curvature**.[2]

I chose a ratio of 2 to 1, adding two new stitches for each previous stitch. With this ratio, my flattish surface started looking like a ball very fast. The length of the perimeter doubled for every row. **Thus each new row I crocheted took the same total amount of time that I had already spent on the whole thing.**[3] All the hours I worked on this "brain," I kept thinking: darn, it is so unrewarding to do this. Annoying as it was, the thing that kept me going was my initial decision to continue to use up all the yarn Debbie had given me. **In the end, with this ratio, half the time I worked was spent making the final row**.

Khovanova's last sentence contains the solution to the fruit fly problem, which I will address in a minute. But the first emphasized sentence poses a new slant that I wanted to justify.

Cumulative Time. Suppose T is the time it took Khovanova to crochet her first row, then it took 2T minutes to crochet the second row and $2^{n-1}T$ minutes to crochet the nth row. Let T_n be the total time it took for her to crochet n rows. Then

$$T_n = T + 2T + 2^2T + \ldots + 2^{n-1}T$$

Applying the standard trick used for geometric series, we multiply T_n by 2 and subtract:

$$T_n - 2T_n = T - 2^nT \Rightarrow 2^nT = T_n + T$$

That is, the time to crochet the n+1 row is equal to the time to crochet all the previous n rows *plus the first row again*. Khovanova neglects the extra time to do the first row. Given how large the times grow, it is not a bad approximation to neglect the extra time for the first row.

In fact, let's consider some numbers to get a feel for the situation. I measured the photo above to get that the "ball" was about 3.5" across, which would yield a radius of 7/4". I then measured the photo to estimate the width of a crochet row as 1/4". Making the (unwarranted) assumption that Khovanova began virtually in the center, that would yield 7 rows of yarn. If we also assume she crocheted her first row in T = 30sec = 1/2min, Then it took her $2^7T = 128(1/2) = 64$min = 1hr 4 min to crochet the last row, which was also how long it took her to crochet the previous 6 rows, neglecting the extra 30 sec for the first row.

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1 JOS: The other characterization of a hyperbolic surface is the perimeter of a circle drawn on the surface is greater than $2\pi r$ where r is the radius. That is, the perimeter of a circle in a flat plane grows linearly as the length of the radius, whereas in a hyperbolic surface it grows faster. In Tanya Khovanova's case the length of the rows is growing exponentially with each added row.

2 JOS: This comment about constant curvature will be explained further below.

3 JOS: Notice that Khovanova's statement that amount of time to do a row is proportional to the cumulative time to do all the rows up the then is a characteristic of exponential growth.

Fruit Fly Problem. The answer to the fruit fly problem is the jar is half full at 59 minutes, since the next minute all the flies double, which fills the jar. That is essentially the same reasoning as Khovanova gives in her last sentence.

Constant Curvature. A thorough discussion about constant curvature will have to be postponed for a more mathematical treatment. But at least I can describe a little more clearly the "surface" that relates to Khovanova's crocheted "ball".

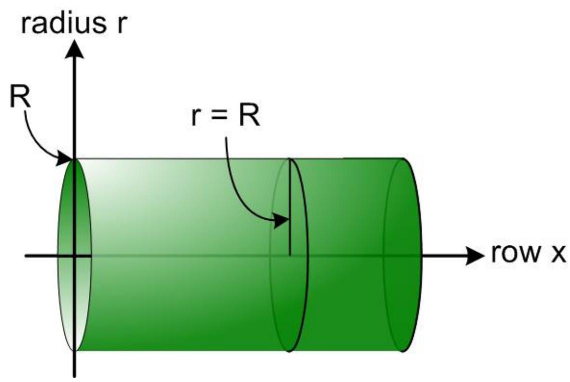

Figure 1 Cylindrical Surface

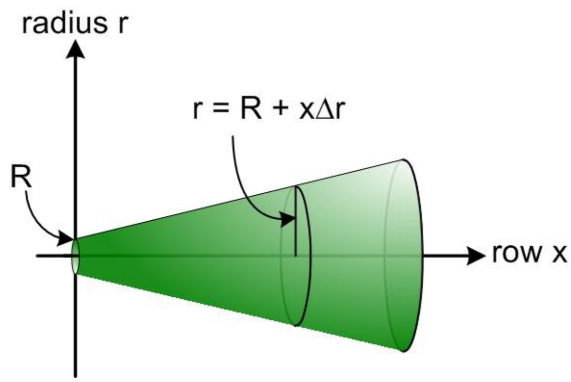

Figure 2 Conical Surface

Let's approach the idea in steps. Suppose Khovanova crotcheted each row in exactly the same length. Then the circumference C of the row would not change and she would get a sleeve (ignoring the first solid disk) as in Figure 1. Note that the radius of each added row would stay constant at r = R, the original size.

Now suppose she crocheted each successive row by adding a few links of length ΔC. If we designate the row number by x, then the circumference of row x would be $C = C_0 + x\ \Delta C$ where $C_0 = 2\pi R$, the original length. (We have "smoothed" the pictures by imagining x changing continuously, rather than in discrete integers.) Now the constant change in circumference corresponds to a constant change in radius $\Delta r = \Delta C/2\pi$. So $C = 2\pi(R + x\Delta r)$ and $r = R + x\Delta r$, as in Figure 2. Since Δr is a constant increment, C increases linearly with each row x.

Now we consider Khovanova's original crocheted "surface." If you pull the original central disk out until the work is stretched full length, it will look something like the picture in Figure 3, where we have "smoothed" the discrete rows into a continuous shape. The idea that each circular row is twice the length of the previous row is equivalent to saying each corresponding radius of the circle is twice the previous. So starting with a disk of radius R, after crocheting x rows, the radius of that row will be $R2^x$. (The picture is not to scale. A picture of true exponential growth takes up too much space.)

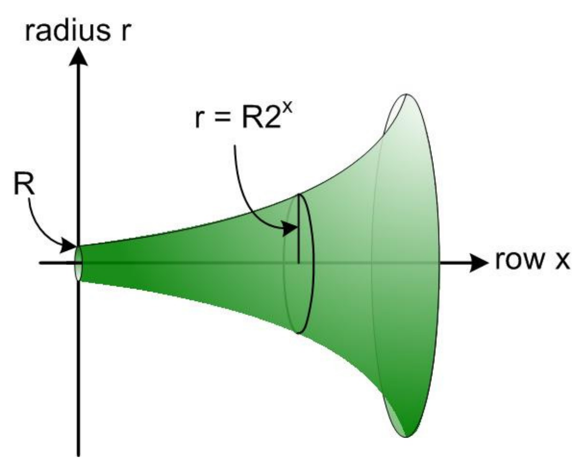

Figure 3 Crocheted Surface

This "bugle"-shaped surface is naturally often called a bugle surface. The exploration of this type of surface will have to wait for a more mathematical analysis that I carried out later. At which point I discovered that the presentation given here included a subtle misconception that confused me for some time.