Two Pints of Cider

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(https://blogs.wsj.com/puzzle/2017/11/16/varsity-math-week-114/, retrieved 11/16/2017)

Photo: Luci Gutierrez

Team member Janice recently visited the U.K. and poses this puzzle to her teammates: You have three containers that can hold exactly 15, 10 and 6 pints. The 15-pint container starts full of cider. You want to measure out exactly 2 pints of cider, drink it all, and end with an empty 15-pint container and 8 and 5 pints of cider in the other two containers.

What transfers should you make to accomplish this?

My Solution

This is a variant of the Three Jugs Problem I considered before ([1]) and is solved with the billiard table method. We begin by constructing an equilateral triangle whose sides are of length the total amount of liquid available in the problem, in this case, 15 pints of cider. Given that all our operations will involve integral values of pints of cider, we construct a grid of three sets of lines where each set consists of lines parallel to the sides of the triangle spaced one pint apart (see Figure 2). Each point of intersection represents a state of the three jugs, that is, a distribution of cider among them. For example, the point (1, 4, 3) represents 1 pint in the 15 pint jug, 4 pints in the 10 pint jug, and 3 pints in the 6 pint jug, where we have color-coded the jugs as red, blue, and green respectively.

Notice that the grid defines a pattern of small triangles 1 pint on a side, as shown in Figure 1. Each vertex of the triangle represents a state of the three jugs where two of the jugs differ from their neighbors by 1 pint. That is, moving from one vertex to another represents pouring 1 pint of cider from one jug into another, thus one jug looses a pint and another gains a pint where the jug represented by the edge connecting the vertices does not change. The arrows in the figure indicate the direction of increase for each of the jugs.

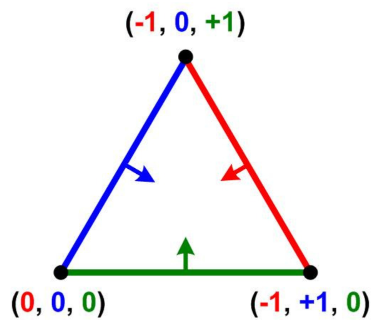


Figure 1 One Pint Pour

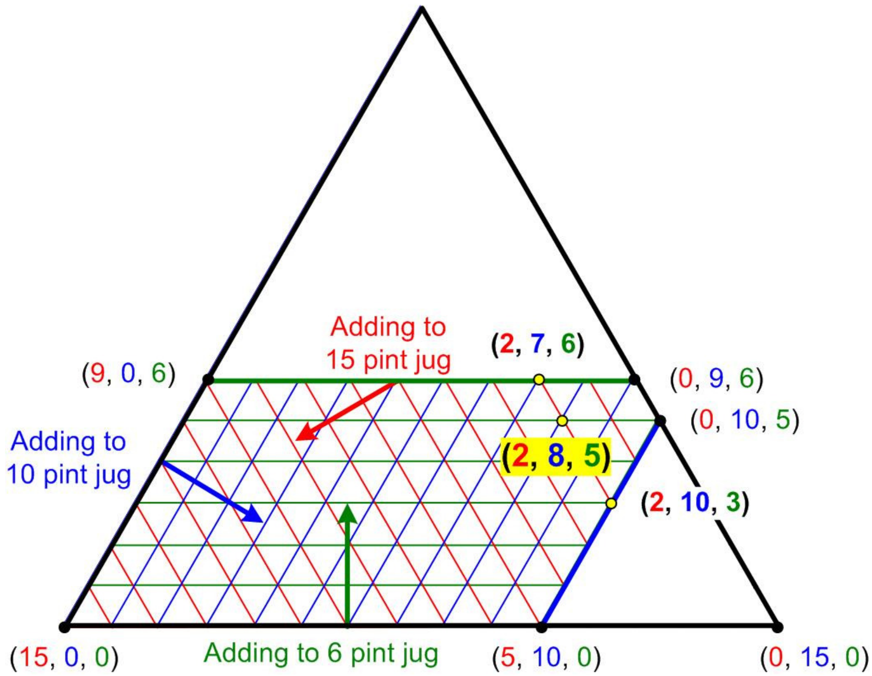


Figure 2 Set up

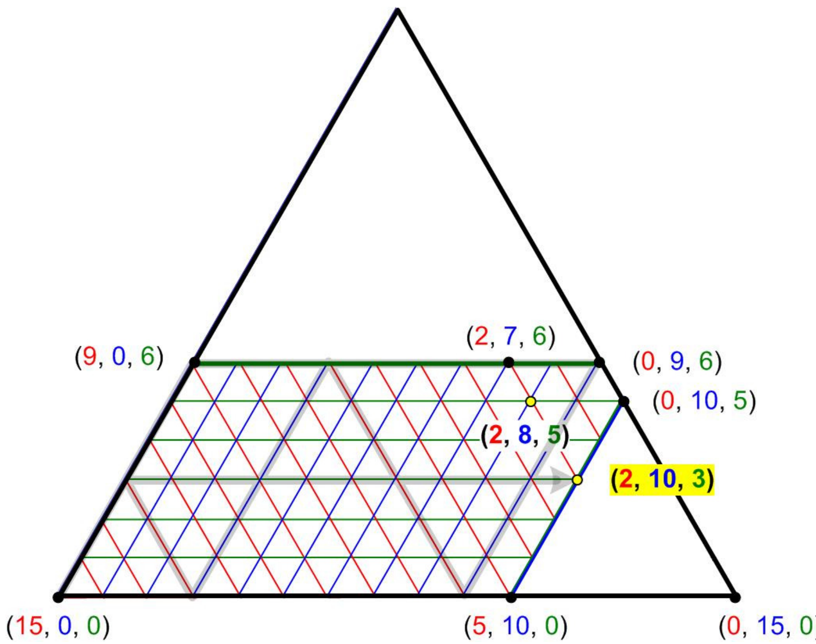
Now the problem is we cannot in general pour one pint from one jug to another. We are constrained to only be able to *empty* a jug into another or *fill* the other jug. This corresponds to moves along the triangle grid to the boundaries of the allowable region of states. The allowable region of states is determined by the fact that we cannot have more than 15 pints of cider in the 15 pint jug, or 10 pints in the 10 pint jug, or 6 pints in the 6 pint jug. This makes the allowable region of states appear like a truncated parallelogram. So every pour takes us from one edge to another of the parallelogram like a bouncing billiard ball (since the 60 degree angles between the lines act as paths of reflection off the edges).

We begin at the state (15, 0, 0) (lower left-hand vertex of the parallelogram) with 15 pints of cider in the 15 pint jug and nothing in the others. We want to end at the state (0, 8, 5) with 8 pints in the 10 pint jug and 5 pints in the 6 pint jug and with an empty 15 pint jug, after we have drunk 2 pints of cider somewhere along the line. If we could get to state (2, 8, 5), then we could drink the 2 pints in the 15 pint jug and be done. But this point is an interior point of the parallelogram and cannot be reached by the allowable pourings. Possibly we can get to either of the states (2, 7, 6) or (2, 10, 3), drink the 2 pints, and then somehow proceed onto the desired state.

Figure 3 shows just such a path of pourings to reach (2, 10, 3) (in 7 steps). We drink the 2 pints and now have a state (0, 10, 3). We are going to write the states for the 15 pint jug in black from now on, since the situation has changed, that is, the total amount of cider available is now 13 pints instead of 15 pints. This means we need to represent the states in an equilateral triangle of sides 13, as in Figure 4.

Figure 4 shows that starting at (0, 10, 3), we can trace a path to the final solution (0, 8, 5) (in 10 steps).

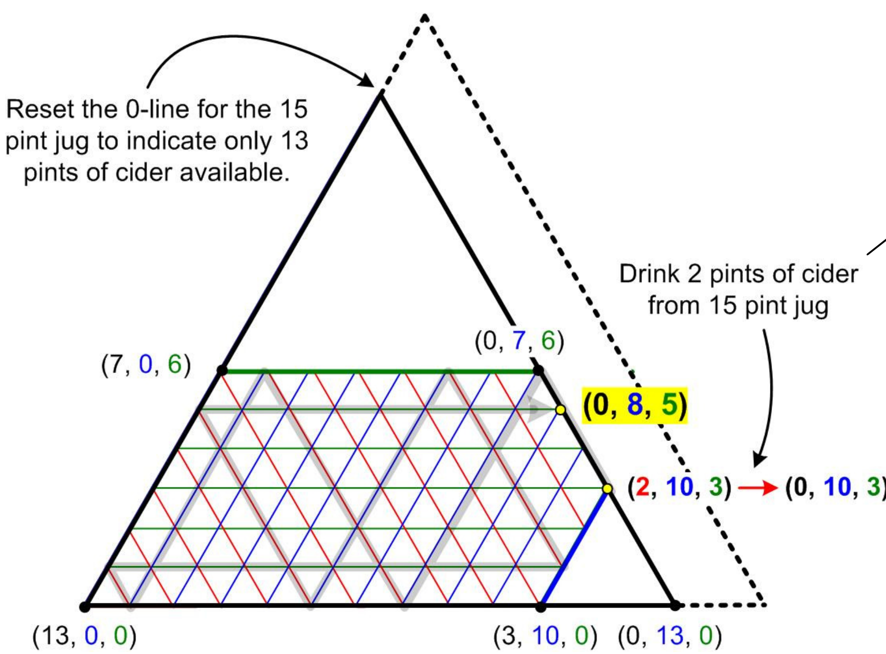
It should be noted that there are other solutions to this problem, and the comments to the original post provided such a different set of pourings (which was actually shorter: 15 steps vs. my 17).



Pouring Sequence

15	10	6
15	0	0
9	0	6
0	9	6
6	9	0
6	3	6
12	3	0
12	0	3
2	10	3

Figure 3 First Stage



Pouring Sequence

15	10	6
2	10	3
0	10	3
0	7	6
6	7	0
6	1	6
12	1	0
12	0	1
2	10	1
2	5	6
8	5	0
8	0	5
0	8	5

Figure 4 Final Stage

References

1. Stevenson, James, "Three Jugs Problem," *Meditations on Mathematics*, 4 August 2013