Triangular Boundary

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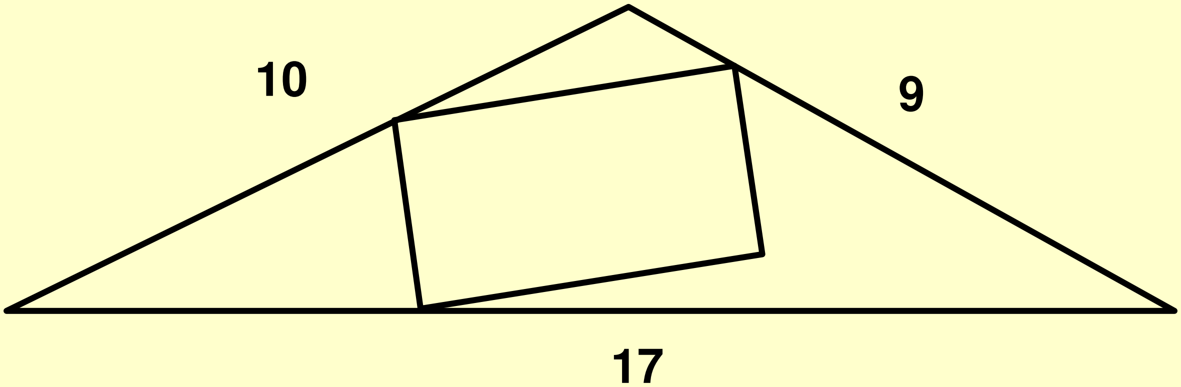
(https://blogs.wsj.com/puzzle/2017/09/29/varsity-math-week-107/, retrieved 10/1/2017)

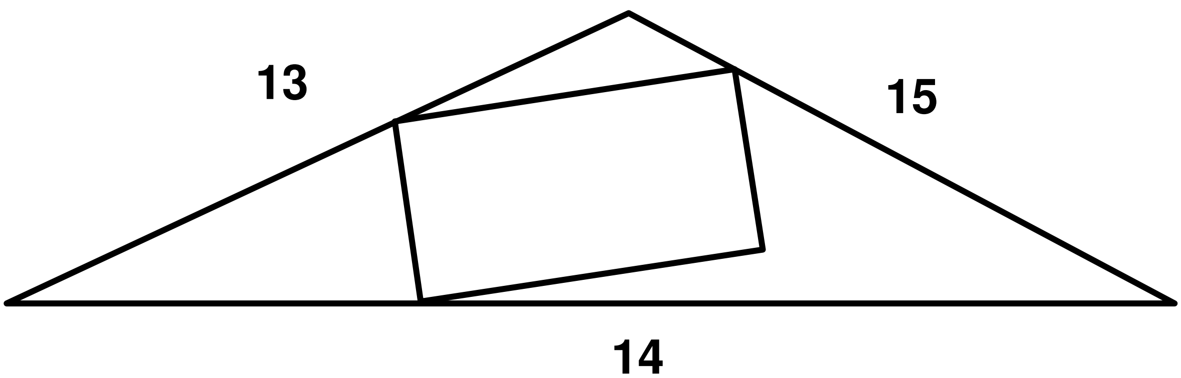
Wall Street Journal Varsity Math Week 107, Sep 29, 2017 12:26 pm ET

The coach then shows the team the diagram below and asks: What is the maximum area of a rectangle contained entirely within a triangle with sides of 9, 10 and 17?



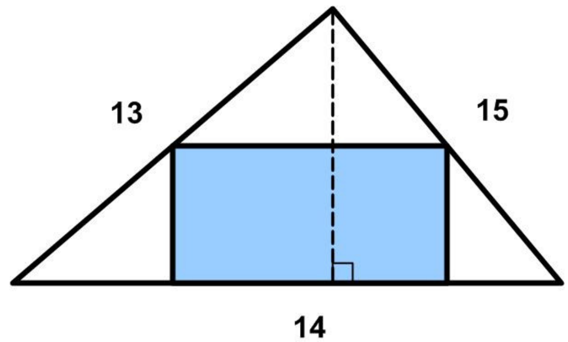
My Version

These numbers provide a cleaner solution, but the idea is basically the same.

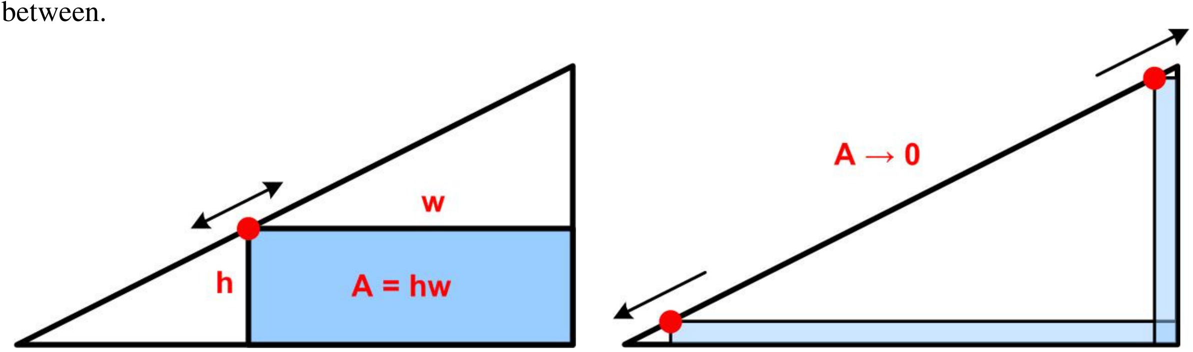


First, notice that any rectangle inscribed at an angle, as shown in the picture, can be rotated to lie flat on the base and so have the same area. But then it can be enlarged until it touches the other side of the triangle. Therefore, we are only interested in rectangles inscribed in the triangle that lie on the base.

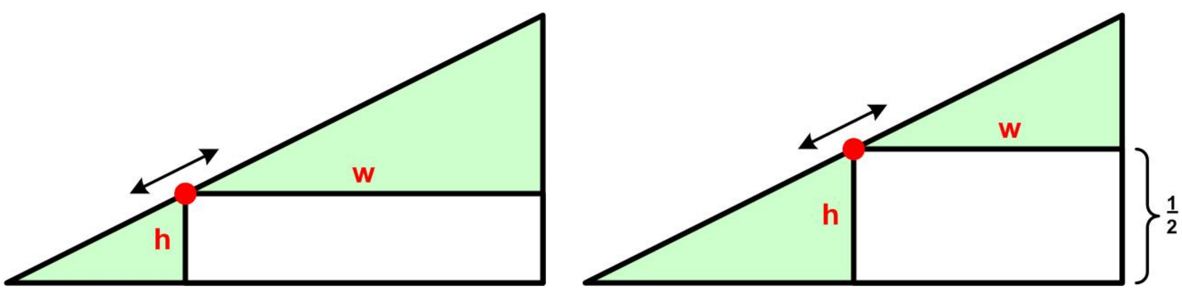
Second, notice that if we drop a perpendicular from the top vertex to the base (forming the altitude of the triangle) that this divides the triangle into two right triangles and also divides the rectangle into two rectangles inscribed in the right triangles. Now maximizing the areas of the two divided rectangles separately may not be the same as maximizing the area of the full rectangle, since the individual rectangles do not have both top vertices constrained, but only one of them. Nevertheless, we shall first see about maximizing the two separate rectangles.



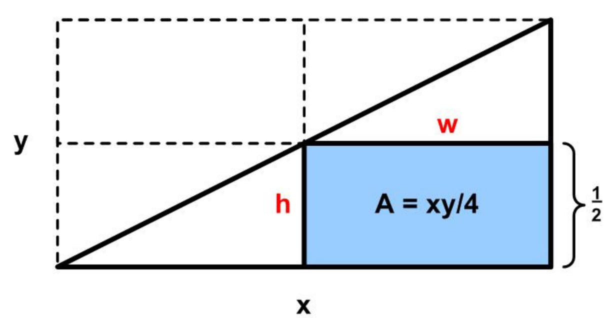
As the red vertex moves along the hypotenuse of the right triangle to either end, we can see that the area of the inscribed rectangle shrinks to 0. Therefore the area must be maximal somewhere in between.



Consider now the area not in the rectangle, namely, the (green) triangular areas. Whenever the inscribed rectangle has its (red) vertex near one of the ends of the hypotenuse, the green triangles are of unequal area. By swapping the two green triangles we have a symmetric case for the inscribed rectangle that yields the same area. Therefore as the red vertex of the rectangle moves away from the ends of the hypotenuse, its area increases and the green triangles' areas decrease. By symmetry the green triangles will add up to a minimum area when they are equal, that is, their sides are one-half the sides of the large right triangle. (This can also be verified by calculus rather easily. See below, p.3.)

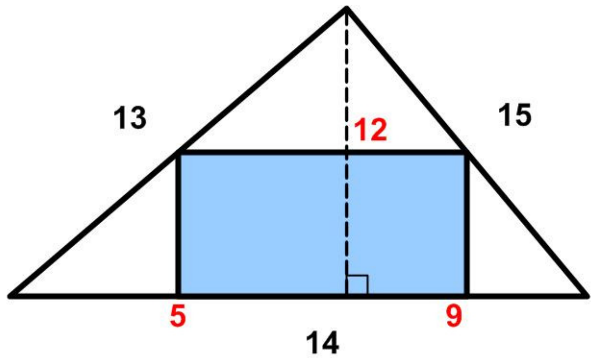


When this occurs, the area of the inscribed rectangle is ¼ the area of the circumscribed rectangle (or ½ the area of the original triangle).



Since this argument holds for any rectangle inscribed in a right triangle as shown, it holds for the two right triangles in our original figure. And since both right triangles have the same altitude, both rectangles have the same height and join into one large rectangle with the common height equal to one-half the altitude of the original triangle.

Since the altitude must be the same for both half right triangles, we have the following two Pythagorean triples for the triangle, namely, (5, 12, 13) and (9, 12, 15), the latter being 3 times the (3, 4, 5) triple.

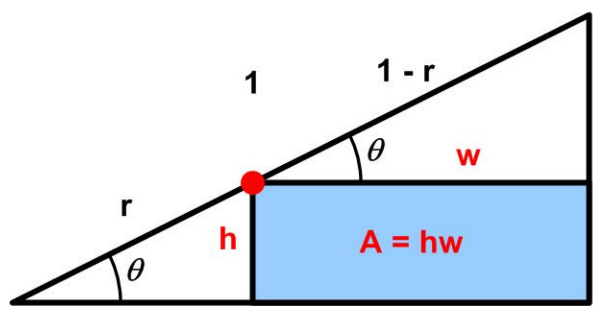


Therefore the altitude of the triangle is 12 and its base 14, which implies that the area of the inscribed rectangle is A = 12 x 14/4 = 42.

(I think the fact that the original triangle has sides of consecutive integers 13, 14, 15 is rather elegant, and the fact that the altitude is another integer in the sequence is even more fun. I don't know why the *Wall Street Journal* picked such ugly numbers for its formulation of the problem. Ah, now I do – see solution to the originally-stated problem below, p.3)

Addendum: Calculus Solution for Maximum Rectangle

Without loss of generality we may assume the length of the hypotenuse is 1. The right triangle will then be determined by the angle θ. Then we have the following values:



$h = r \sin \theta$

$w = (1 - r) \cos \theta$

$A = r(1 - r) \sin \theta \cos \theta$

$= ½ (r - r^2) \sin 2\theta$

$A' = ½ \sin 2\theta (1 - 2r) = 0 \Rightarrow r = ½$.. $A'' = -\sin 2\theta < 0 \Rightarrow r = ½$ is a maximum point for A.

Solution to Original Problem

The difficulty I encountered trying to solve the original problem was in figuring out what the altitude of the triangle was in order to compute its area (or that of the circumscribing rectangle) given only the lengths of its sides. The algebra entailed in applying the Pythagorean Theorem multiple times was excruciating. So I changed the problem to make finding the altitude easier.

But just recently I was reminded of Heron's formula[1] for finding the area A of a triangle given only its sides, say of length a, b, and c:

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[1] *Wikipedia*: The formula is credited to Heron (or Hero) of Alexandria [10 - 70 CE], and a proof can be found in his book, *Metrica*, written c. CE 60. It has been suggested that Archimedes knew the formula over two

$$A = \sqrt{\frac{(a+b+c)}{2}\frac{(a+b-c)}{2}\frac{(a-b+c)}{2}\frac{(-a+b+c)}{2}}$$ **Heron's Formula**

(The derivation of this formula actually involves the multiple applications of the Pythagorean Theorem and the nasty algebra that I wanted to avoid.) So plugging in the given sides of the triangle 9, 10, and 17, yields

$$\begin{aligned}
A^2 &= \frac{(9 + 10 + 17)}{2}\frac{(9 + 10 - 17)}{2}\frac{(9 - 10 + 17)}{2}\frac{(-9 + 10 + 17)}{2} \\
&= \frac{36}{2} \quad \frac{2}{2} \quad \frac{16}{2} \quad \frac{18}{2} \\
&= 18 \quad 1 \quad 8 \quad 9 \\
&= 4^2\, 9^2
\end{aligned}$$

Therefore A = 36 and the area of the largest inscribed rectangle (following my original reasoning) is one-half that or 18.

By the way, applying Heron's formula to my problem statement with sides of the triangle given by 13, 14, and 15, yields

$$\begin{aligned}
A^2 &= \frac{(13 + 14 + 15)}{2}\frac{(13 + 14 - 15)}{2}\frac{(13 - 14 + 15)}{2}\frac{(-13 + 14 + 15)}{2} \\
&= \frac{42}{2} \quad \frac{12}{2} \quad \frac{14}{2} \quad \frac{16}{2} \\
&= 21 \quad 6 \quad 7 \quad 8 \\
&= 3^2\, 4^2\, 7^2
\end{aligned}$$

So that the area of the triangle is A = 3•28, and the area of the maximum rectangle is half that, or 3•14 = 42. This is the answer we got before.

Somehow applying an obscure formula to solve the problem rather than the more straight-forward and well-known definition of the area of a triangle is a bit unsatisfying. Anyway, I think the heart of the problem is figuring out that the maximum inscribed rectangle would be one-half the area of the triangle.

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centuries earlier, and since *Metrica* is a collection of the mathematical knowledge available in the ancient world, it is possible that the formula predates the reference given in that work. JOS: There are multiple equivalent expressions for Heron's formula, such as those involving the *semi-perimeter* s = (a + b + c)/2, namely $A^2 = s(s - a)(s - b)(s - c)$. For the original triangle this is $A^2 = 18{\cdot}9{\cdot}8{\cdot}1 = 36^2$ as before.