Three Jugs Problem Redux

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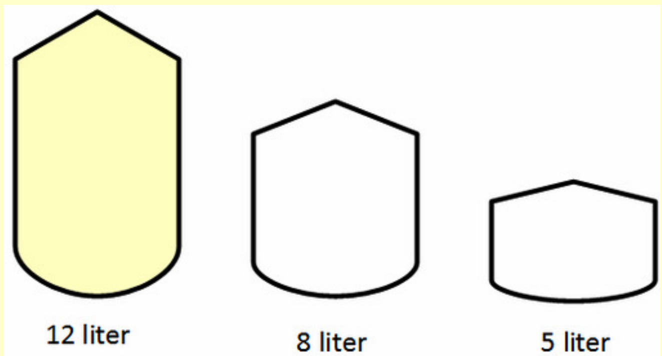
I was sifting back through some problems posed by Presh Talwalkar on his website Mind Your Decisions,[1] when I found another 3 Jugs problem, which was amenable to the skew billiard table solution

(https://mindyourdecisions.com/blog/2015/09/13/the-3-jug-riddle-sunday-puzzle/, retrieved 1/24/2019)

The 3 Jug Riddle – Sunday Puzzle

Posted September 13, 2015 By Presh Talwalkar.



The movie *Die Hard* contained several interesting puzzles, including the water jug riddle.[2] Today's problem is a classic version that appeared in a 1484 book and so delighted a young Poisson that he pursued mathematics. A milkman carries a full 12-liter container. He needs to deliver exactly 6 liters to a customer who only has 8-liter and a 5-liter containers. How can he do this? No milk should be wasted: the milkman needs to leave with 6 liters of milk. Can he measure all amounts of milk from 1 to 12 (whole numbers) in some container?

Solution

Here's how it's done, and how to measure all whole amounts. Write (a, b, c) = (amount in 12 liter, amount in 8 liter, amount in 5 liter) containers. We start out with (12, 0, 0) and we want to end up with (6, 6, 0). Here is how it can be done.

(12, 0, 0)
(4, 8, 0) – pour from 12 into 8
(4, 3, 5) – pour from 8 into 5
(9, 3, 0) – pour from 5 into 12
(9, 0, 3) – pour from 8 into 5
(1, 8, 3) – pour from 12 into 8
(1, 6, 5) – pour from 8 into 5
(6, 6, 0) – pour from 5 into 12

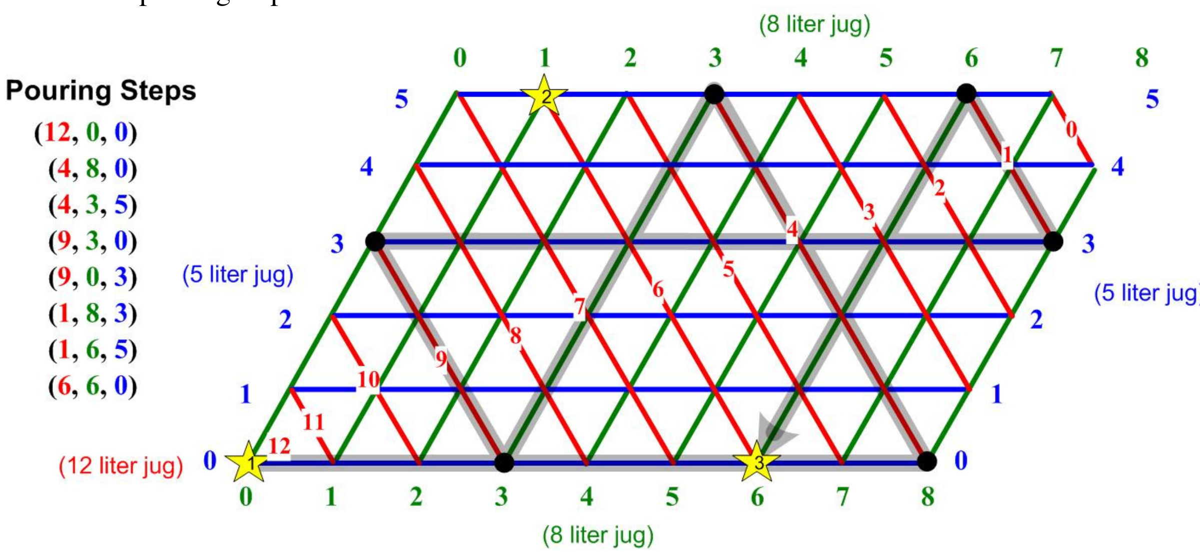
Source: Petkovic, Miodrag S., *Famous Puzzles of Great Mathematicians*, New ed., American Mathematical Society; 2009.

[1] https://mindyourdecisions.com/blog/

[2] https://mindyourdecisions.com/blog/2013/02/04/the-water-jug-riddle/#.VdJUbvlVhBc

Justifying the Solution

Talwalkar does not show how he got the solution, so I thought I would add truncated skew billiard table solution based on the explanation I gave for the Three Jugs Problem. The figure shows there are actually two solutions that leave the customer with 6 liters of milk. The solution marked with the star numbered 2, leaves the 8 liter jug with 1 liter of milk and the 5 liter jug full with 5 liters of milk. The second solution, marked with the star numbered 3 and which agrees with Talwalkar's, leaves all 6 liters of milk in the 8 liter jug. I only showed the second solution, since it was arrived at with fewer pouring steps.



Finding Other Distributions Of Milk

Then Talwalkar goes on to discuss how other amounts of milk could be obtained, which basically just seemed to be values of the coordinates of the points on the boundary that were reached by the "ricocheting billiard ball." He then said

In problems like this, we can figure out if it's possible to measure an amount even before we explicitly come up with a method. If we have jugs that hold *a* liters and *b* liters, then transferring water between them corresponds to the linear Diophantine equation $ax + by = c$ where *x* and *y* are integers [?]. The theory of Diophantine equations states you can create an amount *c* if and only if it is a multiple of the greatest common divisor of *a* and *b*. (proof[3]) The problem above is slightly different as we have 3 numbers, but the same idea holds. **Since 1 is the greatest common factor of 12, 8 and 5, we can measure all liters from 1 to 12 (whole numbers).**[4]

I found this discussion quite confusing. Was the c of the Diophantine equation the third jug? The later sentences about 3 numbers instead of 2 seemed to suggest no. So was the c some amount of milk that the a and b jugs would hold at some point? None of the details or connection with the 3 jugs problem were really made clear, nor how the values of x, y in the Diophantine equation related to the pouring steps, but I think I have finally figured out what Talwalkar was trying to get at.

The value c *is* the value of the amount of milk in one of the two jugs holding a and b liters. The values x and y have nothing to do with the pouring steps as far as I could determine. All that matters is the gcd property. Extending the explanation to three jugs is not difficult. What Talwalkar is saying is that the only amounts of milk that the three jugs can each hold must be divisible by the greatest

[3] http://mathforum.org/library/drmath/view/51595.html

[4] JOS: See my caveat below p.4

common denominator of the three sizes of the jugs. If the three jugs are relatively prime (the gcd = 1), then any value from 0 to the size of the largest jug is possible in some jug.

The easiest way to see what is going on is to look at some examples. Assume first that the largest jug size equals the sum of the smaller jug sizes. Notice this means any whole number that divides both of the smaller jugs will therefore divide the largest. Since the greatest common divisor of the three numbers can be obtained though a succession of pairwise gcd's (gcd(a, b, c) = gcd(gcd(a, b), c)), if a + b = c, then gcd(a, b, c) = gcd(gcd(a, b), c) = gcd(a, b)

Figure 1 shows a case where the smaller jug size divides the next larger jug size (and therefore also the largest jug size). Namely, we have 4 liter, 8 liter, and 12 liter jugs. The pouring path skips through on 4 liter intervals hitting 0, 4 in the 4 liter jug, 0, 4, 8 in the 8 liter jug, and 0, 4, 8, 12 in the 12 liter jug. Now 4 = gcd(4, 8) = gcd(4, 8, 12), so only multiples of 4 are possible, as we see.

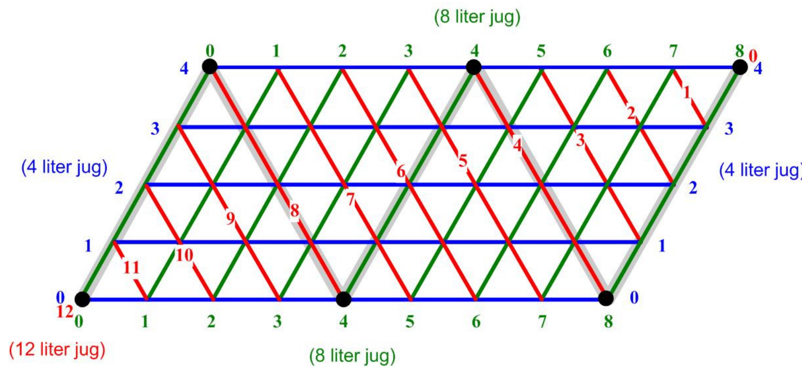


Figure 1 12-8-4 Jug Scenario

Figure 2 illustrates the case of 4 liter, 6 liter, and 10 liter jugs. Here 2 = gcd (4, 6) = gcd (4, 6, 10) and we see the pouring path skips the odd amounts and only yields the even amounts in each of the jugs, namely, 0, 2, 4 in the 4 liter jug, 0, 2, 4, 6 in the 6 liter jug, and 0, 2, 4, 6, 8, 10 in the 10 liter jug.

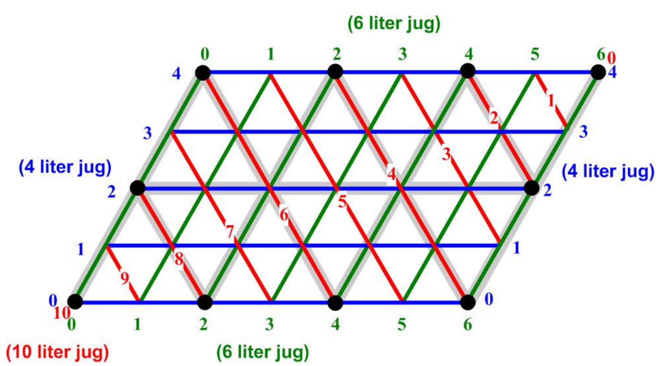


Figure 2 10-6-4 Jug Scenario

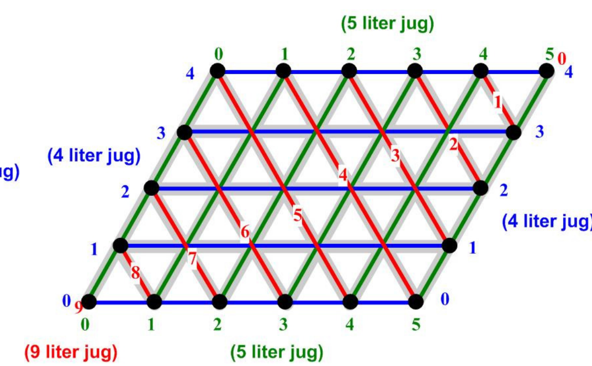


Figure 3 9-5-4 Jug Scenario

Figure 3 shows the case where all the sizes are relatively prime with 4, 5, and 9 liter jugs. Thus 1 = gcd(4, 5) = gcd(4, 5, 9) and so we get all the amounts up to the largest jug size in the 3 jugs.

Things become interesting when we relax the condition that the largest jug size equal the sum of the other two. First we consider a case where the largest jug size is one more than the sum of the smaller ones, namely, 4 liter, 6 liter and 11 liter jugs. This situation is shown in Figure 4. Now gcd(gcd(4, 6), 11) = gcd(2, 11) = 1. So all three numbers are jointly relatively prime. But there is a difficulty. Since 4 and 6 are not relatively prime, the pouring path still skips through their even values, but in each case the largest jug holds an odd valued amount of milk. However, due to the maximum value of 6 for the larger of the smaller jugs, the amounts 8 and 10 are not found in any of the jugs. This is at variance with Talwalkar's implied statement from the case of 5 liter, 8 liter, and 12 liter jugs that because 1 = gcd(5, 8, 12) = gcd(gcd(5, 8), 12) = gcd(1, 12), all the values 0, 1, 2, …, 12 should be found in one or more jugs.

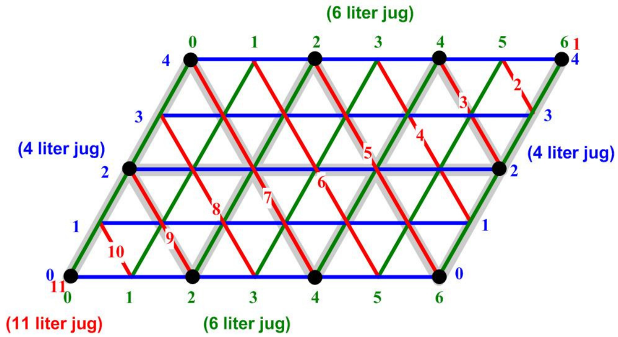


Figure 4 11-6-4 Jug Scenario

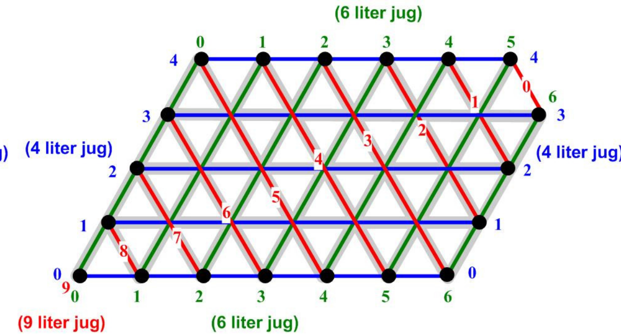


Figure 5 9-6-4 Jug Scenario

Perhaps an insight into the problem can be seen in Figure 5. This is another case where the size of the largest jug does not equal the sum of the two smaller ones. Only in this case, the size of the largest jug is smaller than the sum. We have 4 liter, 6 liter, and 9 liter jugs. Even though gcd(4, 6) = 2 and gcd(4, 6, 9) = gcd(gcd(4, 6), 9) = gcd(2, 9) = 1, this time all the amounts from 0 to 9 are found. This appears to reflect the situation in Talwalkar's case, namely, the largest jug size is smaller than the sum of the other two. Of course, it still must be jointly relatively prime, otherwise there will be missing values again.

So at first glance it looks like *if* the two smaller jugs to be relatively prime (gcd(a, b) = 1), then the three jugs will also be relatively prime and all the values from 0 to the size of the largest jug will be reached (with one final caveat: the largest jug size must not be greater than one more than the sum of the two smaller jugs, due to the subtractive nature of the largest jug amounts). This is a sufficient condition, but not a necessary one, for Figure 5 shows the case where the smaller jugs are not relatively prime, but still all the amounts from 0 to 9 are found.

(Update 1/26/2019) In reviewing the original Three Jugs Problem, I now recognized the name of Burkard Polster, a.k.a. the Mathologer,[5] whose Youtube site has a number of excellent videos explaining mathematics. One of them, How not to Die Hard with Math,[6] posted on 29 May 2015, covers the version of the Three Jugs Problem presented in the movie, *Die Hard*. He also addresses the use of the greatest common divisor.

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[5] https://www.youtube.com/channel/UC1_uAIS3r8Vu6JjXWvastJg/videos

[6] https://www.youtube.com/watch?v=0Oef3MHYEC0