Polygon Areas

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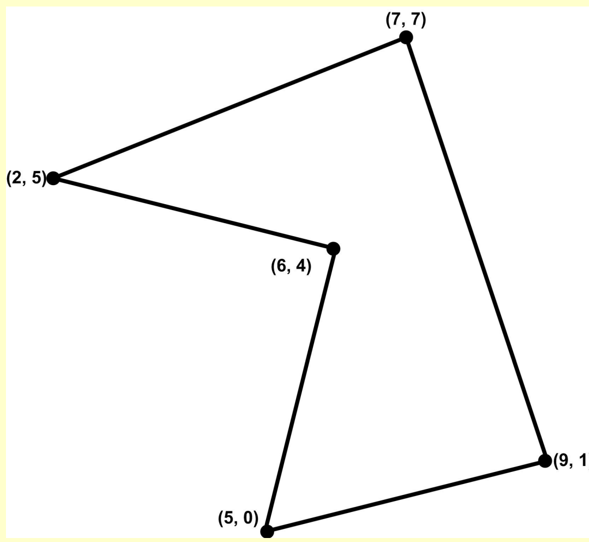
Here is another interesting problem from Futility Closet. They provide the problem with a "solution" but no justification. I thought I would add the explanation.

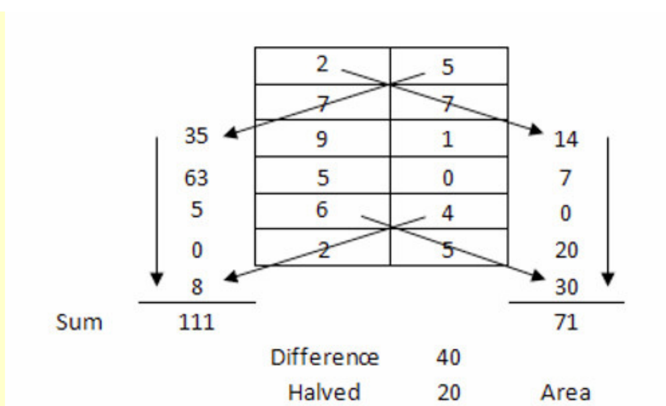
(https://www.futilitycloset.com/2019/01/15/area-matters/, retrieved 1/15/2019)

Area Matters



If you know the vertices of a polygon, here's an interesting way to find its area:

1. Arrange the vertices in a vertical list, repeating the first vertex at the end (see below).
2. Multiply diagonally downward both ways as shown.
3. Add the products on each side.
4. Find the difference of these sums.
5. Halve that difference to get the area.



This works for any polygon, no matter the number of points, so long as it doesn't intersect itself. (Thanks, Derek and Dan.)

Solution.

Recall the definition of the cross product of two vectors u and v:

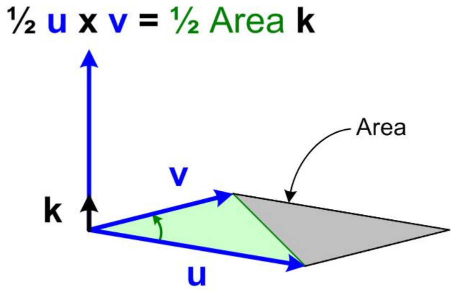


Figure 1 The magnitude of the cross product vector is the area of the parallelogram spanned by the two vectors. So half the vector give half the area or the area of the triangle spanned by the vectors. Since the angle from u to v is positive relative to k, the area is positive.

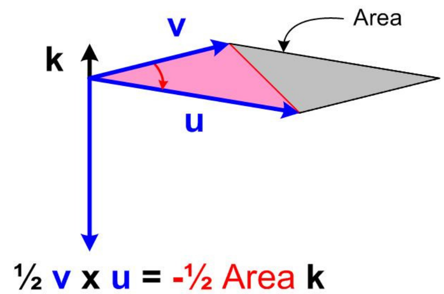


Figure 2 Reversing the order of the cross product of two vectors changes the sign of the resulting vector. The angle from v to u is negative relative to k. This corresponds to the negative of the area spanned by the two vectors.

We draw vectors from the origin to each vertex of the polygon, numbering the points in a counter-clockwise consecutive order: $P_1, P_2, \ldots, P_5$. We designate the vectors as $\mathbf{OP_i}$, i = 1, 2, …, 5. For example, $\mathbf{OP_1} = 2\mathbf{i} + 5\mathbf{j}$. Then the area of the polygon becomes

$$\text{Area of Polygon} = \tfrac{1}{2}\,\mathbf{OP_1 \times OP_2} + \tfrac{1}{2}\,\mathbf{OP_2 \times OP_3} + \ldots + \tfrac{1}{2}\,\mathbf{OP_5 \times OP_1} \qquad (1)$$

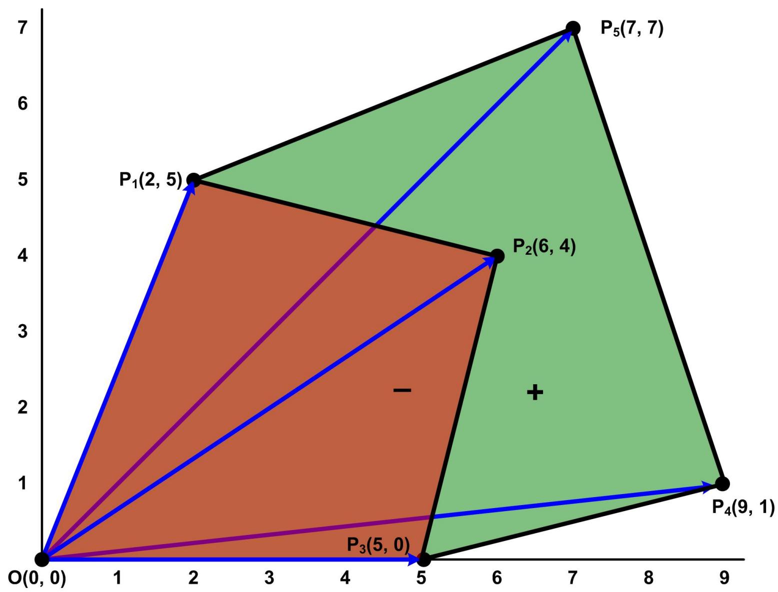


Figure 3 The problem area decomposed into vectors from the origin to each point defining the polygon with the areas from the corresponding cross products (green for positive areas and red for negative areas).

The unit vector **k** is normal to the plane of the polygon and pointing out from the page toward the reader. Since the angle from $\mathbf{OP_1}$ to $\mathbf{OP_2}$ is negative, the cross product $\tfrac{1}{2}\,\mathbf{OP_1 \times OP_2}$ is negative, and since the angle from $\mathbf{OP_3}$ to $\mathbf{OP_4}$ is positive, the cross product $\tfrac{1}{2}\,\mathbf{OP_3 \times OP_4}$ is positive.

Now we consider the effect of the cross products on the components of the vectors, i.e., the coordinates of the vertices of the polygon. We use the determinant form of the cross product. So if $\mathbf{OP_n} = a_n\mathbf{i} + b_n\mathbf{j}$, n = 1, 2, …, 5, then

$$\mathbf{OP_1 \times OP_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \end{vmatrix} = (a_1b_2 - a_2b_1)\,\mathbf{k}$$

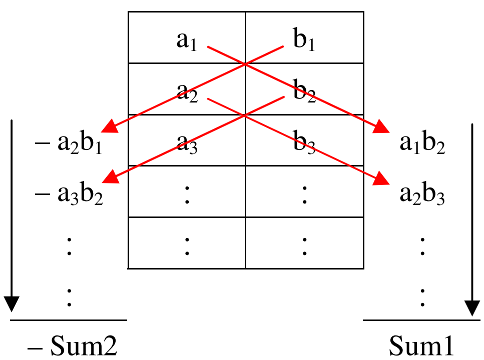
$$\mathbf{OP_2 \times OP_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} = (a_2b_3 - a_3b_2)\,\mathbf{k}$$

$$\cdots$$

Therefore equation (1) becomes

$$\tfrac{1}{2}\left((a_1b_2 - a_2b_1) + (a_2b_3 - a_3b_2) + \ldots\right)\mathbf{k}$$

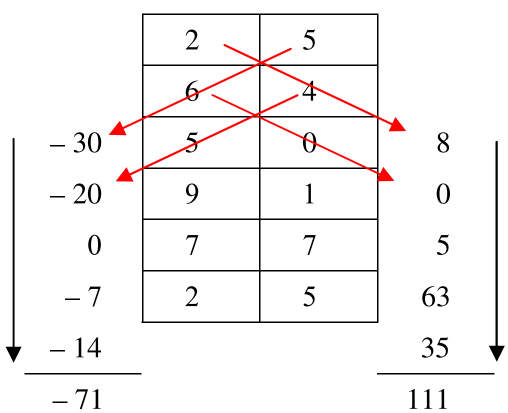
or



so that the area becomes

$$\tfrac{1}{2}\,(\text{Sum1} - \text{Sum2})$$

The Futility Closet solution went around the polygon in the clockwise direction. If we use the counter-clockwise direction as shown in Figure 3, then we have



Area = ½ (111 – 71) = ½ (40) = 20