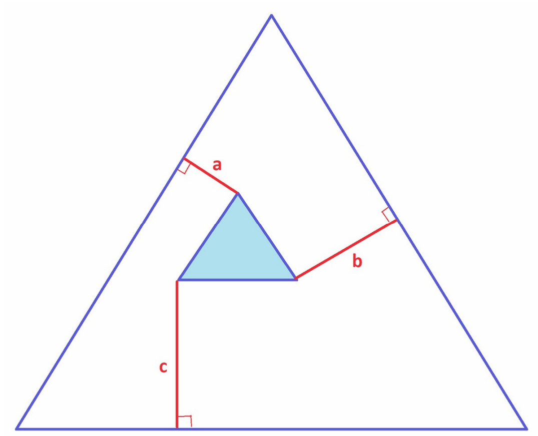
Polygon Altitude Problems I

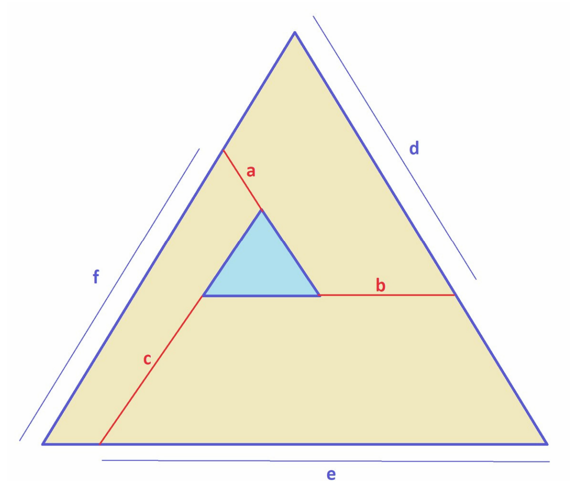
(27 August 2018)

Jim Stevenson

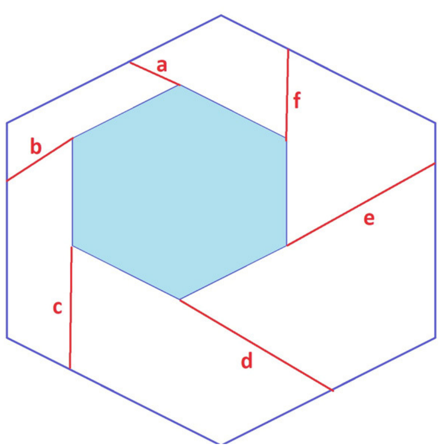
I found the following related problems by James Tanton on Twitter (@jamestanton https://twitter.com/jamestanton).



Problem 1.[1] An equilateral triangle moves about inside another maintaining its matching orientation. What can you say about the sum a + b + c of distances shown?



Problem 2.[2] A small equilateral triangle moves about inside a large one while maintaining its orientation to match that of the large one. What can you say about the sum of distances a + b + c shown? Anything to be said about d + e + f?



Problem 3.[4] A small regular hexagon moves about inside another regular hexagon always keeping its orientation aligned. What can you say about the sum of distances a + b + c + d + e + f shown as it moves?

Problem 4.[3] Classic: Prove that if a convex polygon has rotational symmetry of any degree (eg 60 deg symm, 180 deg symm, etc), then for a point inside the polygon the sum of its distances to each side of the polygon is a fixed predetermined value. (Viviani = 60 deg symmetry.)

Lord Karl Voldevive @Karl4MarioMugan Replying to @jamestanton: This is only valid, if the sides of the polygon are extended, so that each distance meets its line perpendicularly. One example is a long thin rhombus with 180 degree rotational symmetry.

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[1] https://twitter.com/jamestanton/status/1028991577803251714, 13 August 2018
[2] https://twitter.com/jamestanton/status/1029718361687523328, 15 August 2018
[3] https://twitter.com/jamestanton/status/1030126963246153729, 16 August 2018
[4] https://twitter.com/jamestanton/status/1029364678139342848, 14 August 2018

Even though all these problems do not involve perpendiculars, they have a common solution approach. In a later tweet[5] Tanton refers to a Viviani Theorem associated with these types of problems. I do not recall that theorem explicitly or by name. I also have not looked it up yet, in order to solve these problems on my own. I am guessing there is a more classical Euclidean geometry proof, but I like the vector approach for its clarity. (For the Viviani Theorem and Euclidean proof see p.9)

Solution to Problem 1

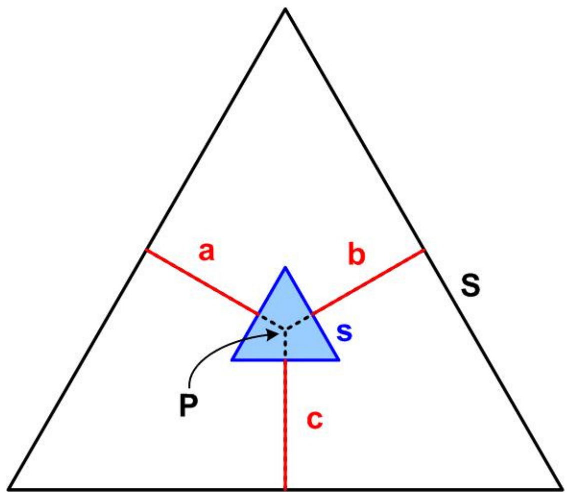

Figure 1

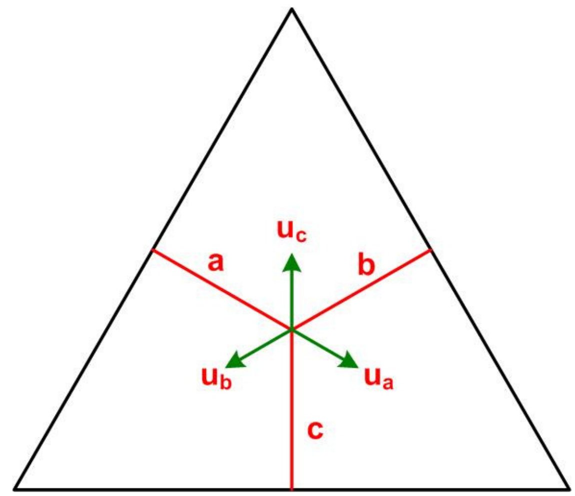

Figure 2

The first thing we do is center the blue triangle. Then we move the perpendiculars so their extensions meet in the center P of the blue triangle as shown in Figure 1. This last operation does not change the lengths of the perpendiculars. We designate the length of the edge of the large triangle as S and that of the small blue triangle s. We shall first solve the problem for the perpendiculars meeting at P in the large triangle. Scaling the result to the small blue triangle and subtracting that from the result for the large triangle gives the answer to the original problem. For ease of use, we shall continue to label the length of the perpendiculars converging on P as a, b, c.

Next we consider unit vectors $\mathbf{u_a}$, $\mathbf{u_b}$, $\mathbf{u_c}$ lying along the perpendiculars and emanating from P as

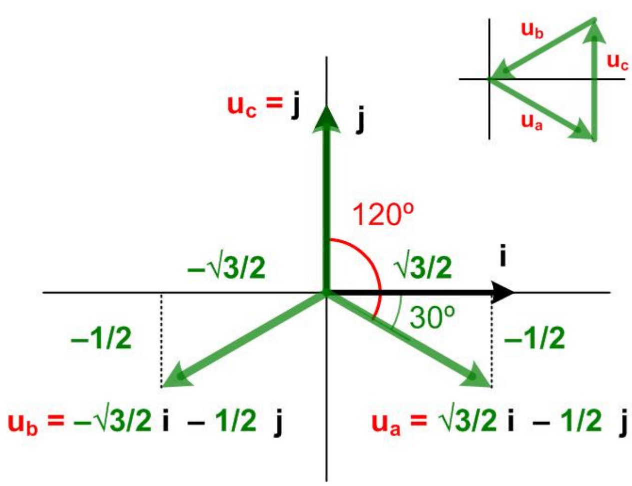

Figure 3 Triangle unit vectors

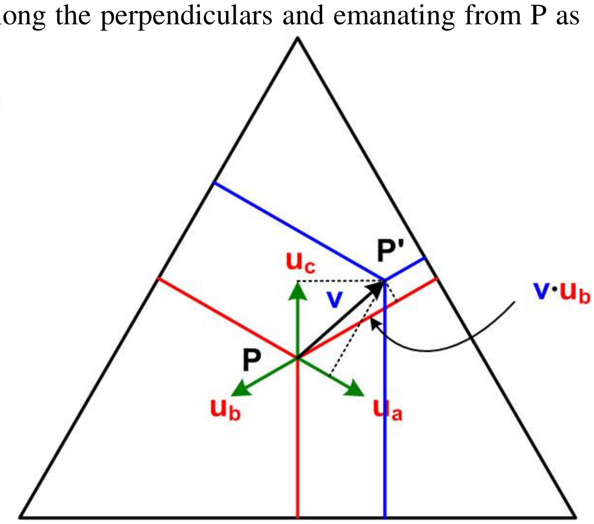

Figure 4 New position for P via vector v

[5] https://twitter.com/jamestanton/status/1030655003881623553, 17 August 2018

shown in Figure 2. Figure 3 shows their values. The idea is to move P (and the perpendiculars) around inside the large triangle to some point P' and see what the effect is on the sum of the lengths. Let **v** be the vector from P to P'. Then the projection of **v** onto each of the unit vectors gives the change in the corresponding lengths of the perpendiculars (see Figure 4). For example, $\mathbf{v} \cdot \mathbf{u}_b$ is the change in the length b (in this case negative). Therefore, we sum the changes to the three perpendiculars:

$$\mathbf{v} \cdot \mathbf{u}_a + \mathbf{v} \cdot \mathbf{u}_b + \mathbf{v} \cdot \mathbf{u}_c = \mathbf{v} \cdot (\mathbf{u}_a + \mathbf{u}_b + \mathbf{u}_c) = \mathbf{v} \cdot ((\sqrt{3}/2\mathbf{i} - 1/2\mathbf{j}) + (-\sqrt{3}/2\mathbf{i} - 1/2\mathbf{j}) + \mathbf{j}) = \mathbf{v} \cdot \mathbf{0} = 0$$

In other words, there is no change to the sum of the lengths no matter where P moves in the triangle. (Notice in the inset to Figure 3 how the unit vectors when added vectorially (tail to head) add to **0**. This is because the equilateral triangle implies the unit vectors are equally spaced at 120° intervals.)

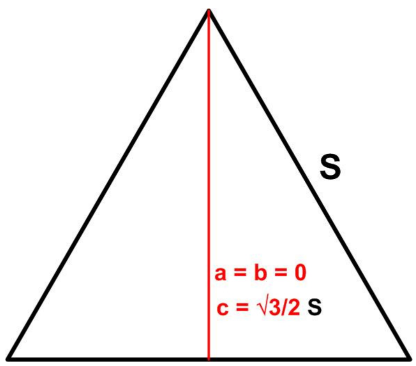

Figure 5 Initial a + b + c Evaluation

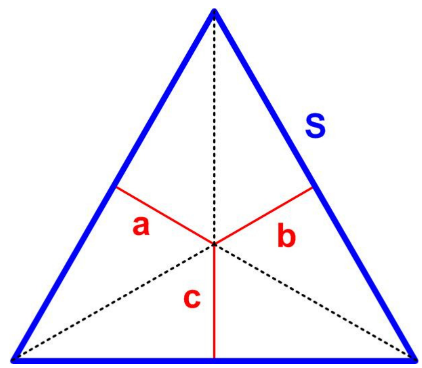

Figure 6 Alternative a + b + c Evaluation

Moving P to the top vertex of the large triangle yields a = b = 0 and c = √3/2 S (Figure 5). Therefore the sum a + b + c = √3/2 S for all points P' in the large triangle. Thus, the corresponding sum for the small blue triangle is a + b + c = √3/2 s. The value of the original sum is then

$$a + b + c = \sqrt{3}/2\,(\mathbf{S} - \mathbf{s})$$

There is an alternative way to compute the constant a + b + c. From the initial central position of the point P we see that the distances a = b = c (Figure 6), again because the triangle is equilateral. Given all the relevant congruent triangles in the equilateral triangle, we see that a / (S/2) = tan 30° = 1/√3 ⇒ a = S / 2√3. Therefore. a + b + c = 3a = √3/2 S, as before.

Solution to Problem 2

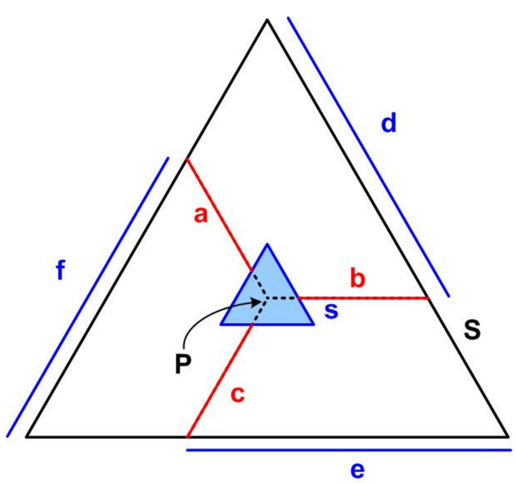

Figure 7

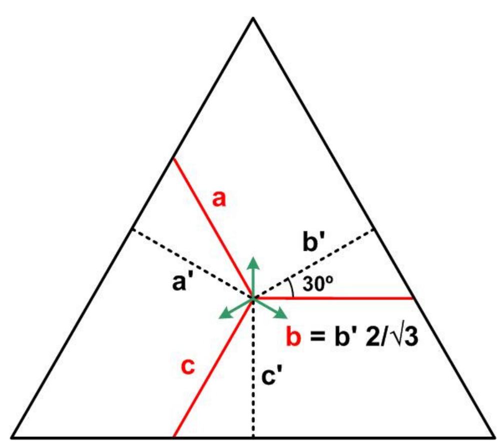

Figure 8

We proceed as in Problem 1 as shown in Figure 7, shifting and extending the line segments to meet in the middle of the centered blue triangle. In order to use the same argument as before, we need to add the perpendiculars from the center (point P) as shown in Figure 8. As also shown in Figure 8, the desired lengths are $2/\sqrt{3}$ times the corresponding perpendicular lengths. Thus, from the solution to Problem 1 we have

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 2/\sqrt{3}\,(a' + b' + c') = 2/\sqrt{3}\,(\sqrt{3}/2\,(\mathbf{S} - \mathbf{s})) = \mathbf{S} - \mathbf{s}$$

The sum d + e + f takes some further reasoning. Figure 9 shows the configuration after the point P has moved off center. The figure shows that the length d is the same as the b line extended to the opposite side of the triangle. Similarly, for the other two lengths. Figure 9 further shows the following relationships

$$a = d - b, \quad b = e - c, \quad c = f - a$$

This means

$$d + e + f = a + (f - a) + b + (d - b) + c + (e - c)$$

$$= a + c + b + a + c + b$$

$$= 2\,(a + b + c) = 2S$$

$$\therefore \quad \mathbf{d} + \mathbf{e} + \mathbf{f} = 2\,\mathbf{S}$$

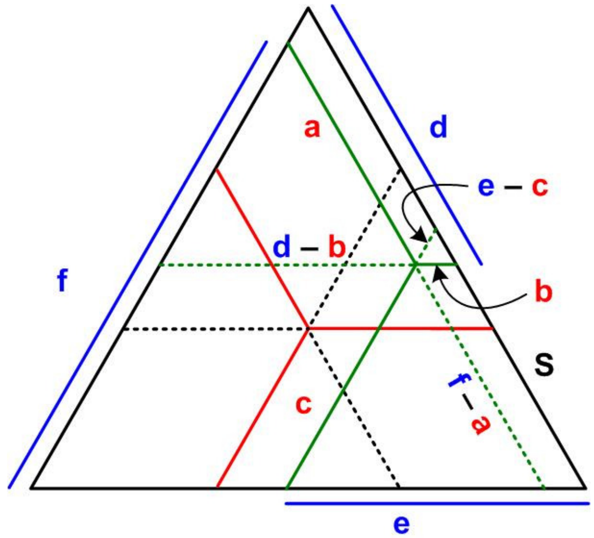

Figure 9

Partial Solution to Problem 3

Perpendiculars Case

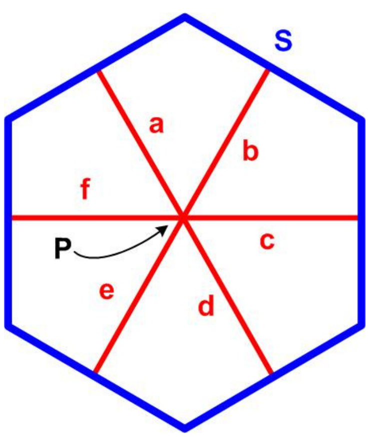

Figure 10

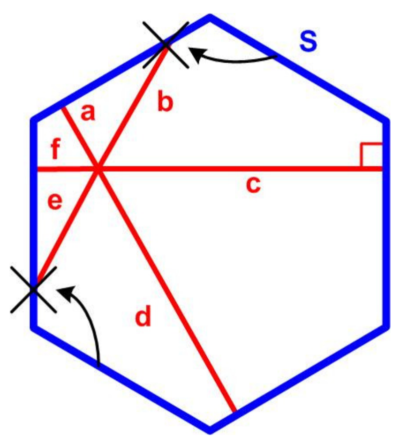

Figure 11

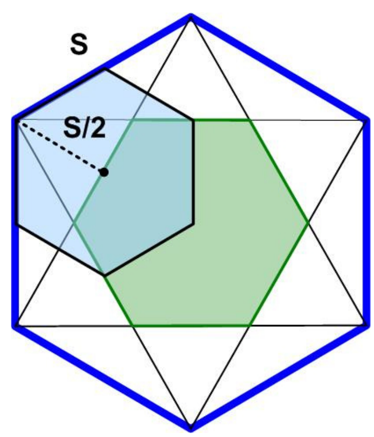

Figure 12

If we proceed as in Problem 1 and consider the perpendiculars all emanating from a point P, starting at the center position (Figure 10), we soon confront the problem shown in Figure 11 where the “perpendiculars” no longer intersect the original side at 90°, but rather an adjacent side at an acute angle.

Figure 12 shows a remedy where a constraint is placed on the smaller blue hexagon moving inside the larger hexagon, namely, restrict its size so that the distance from its center to any vertex is greater than one half the length of the side of the larger hexagon. Another remedy is to apply Karl Mugan's comment to Problem 4, "This is only valid, if the sides of the polygon are extended, so that each distance meets its line perpendicularly."(Figure 13). As in the equilateral triangle case, we consider unit vectors in the direction of increasing length for the perpendiculars (Figure 14). Because the hexagon is regular, each unit vector is 60° apart from its nearest neighbors, and so their sum is again **0** (Figure 15).

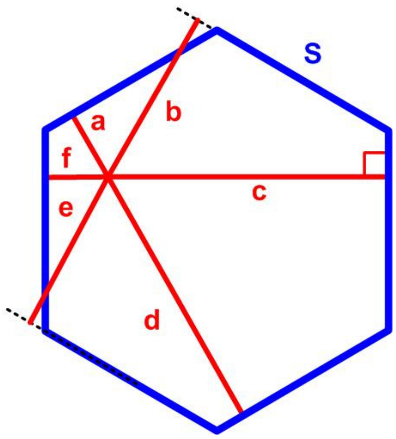

Figure 13 Perpendiculars to extended sides

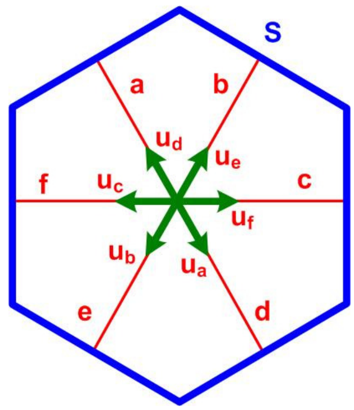

Figure 14 Unit vectors for hexagon perpendiculars

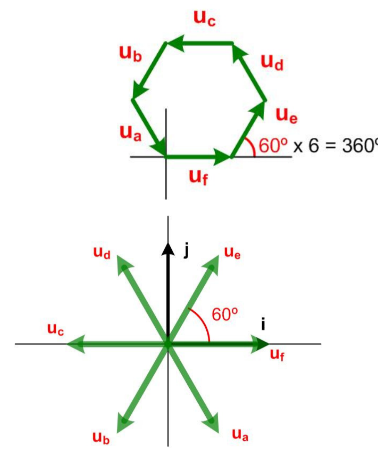

Figure 15 Unit vectors and zero sum

We continue as in the equilateral triangle case (Figure 4) and apply of vector **v** from the center position of P to an arbitrary new position P' inside the hexagon. Projecting **v** onto each of the unit vectors computes the corresponding changes in the distances from the (extended) edges.

$$\mathbf{v}\cdot\mathbf{u_a} + \mathbf{v}\cdot\mathbf{u_b} + \mathbf{v}\cdot\mathbf{u_c} + \mathbf{v}\cdot\mathbf{u_d} + \mathbf{v}\cdot\mathbf{u_e} + \mathbf{v}\cdot\mathbf{u_f} = \mathbf{v}\cdot(\mathbf{u_a} + \mathbf{u_b} + \mathbf{u_c} + \mathbf{u_d} + \mathbf{u_e} + \mathbf{u_f}) = \mathbf{v}\cdot\mathbf{0} = 0$$

Now to find the value of the constant sum for this regular hexagon. We use the second method mentioned for the equilateral triangle. From the center position we have that all the perpendicular distances are equal, that is, a = b = c = d = e = f. Now a / (S/2) = tan 60° = √3 (Figure 16). Therefore, a = 3√3 S, so that

$$\mathbf{a + b + c + d + e + f} = 6\,\mathbf{a} = 3\sqrt{3}\,\mathbf{S}$$

This at least shows a solution for the hexagon with perpendiculars from the center. Having perpendiculars from the corners, as in Problem 1, should just shift the green hexagon allowable region for the center of the small hexagon.

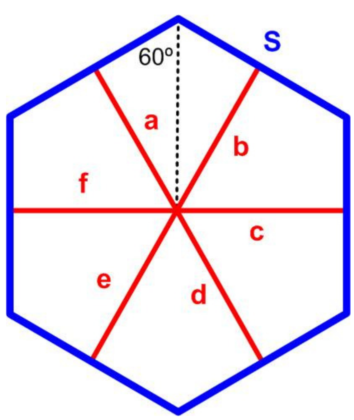

Figure 16 Computing a + b + c + d + e + f

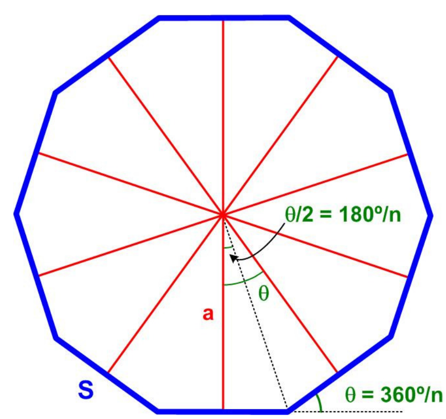

Figure 17 Regular n-sided polygon

It is possible to generalize this approach to any regular n-sided polygon (Figure 17).

$$a_1 + a_2 + \ldots + a_n = n\, a_1 = n\, S/2 / \tan 180°/n \qquad (1)$$

"Slant" Line Case

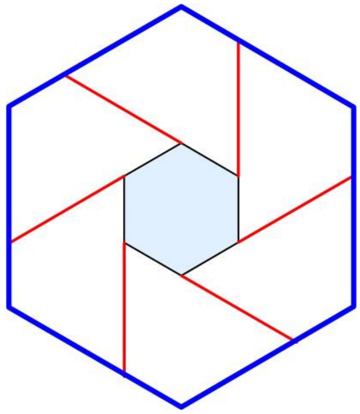

Figure 18

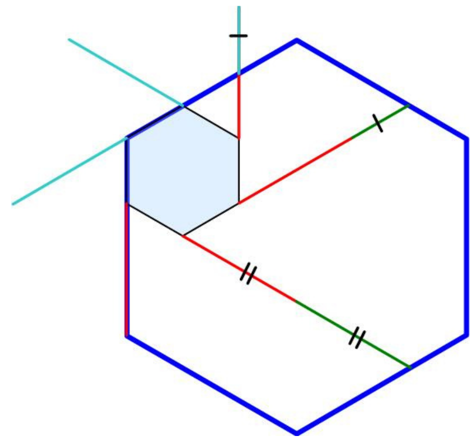

Figure 19

It does not look like the original slant line case of Problem 3 yields the same constant results. Figure 19 shows one example of moving the small hexagon around the inside of the large hexagon starting from the center position (Figure 18). The aqua lines show red lengths lost and the green lines show red lengths gained. The short loss just equals the short gain and so cancels. The long gain just equals one original red length, but two were lost. So that leaves a net one length lost, which implies the sum of the lengths did not remain constant. I am not sure what other relationship might remain invariant.

Solution to Problem 4

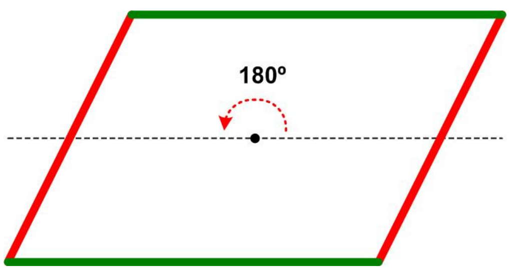

Figure 20 Parallelogram Rotational Symmetry (p = 2)

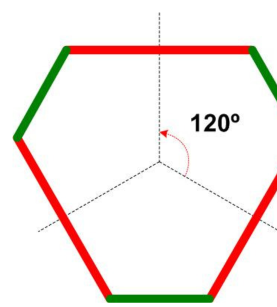

Figure 21 "Diamond" Rotational Symmetry (p = 2)

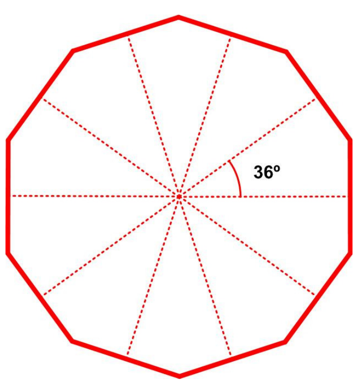

Figure 22 Regular Decagon (p = 1)

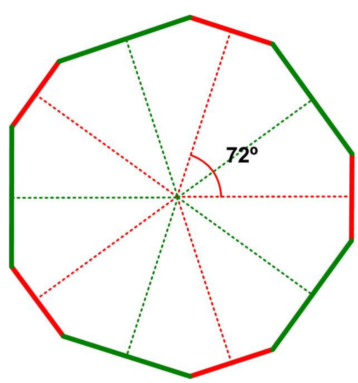

Figure 23 Irregular Decagon (p = 2)

Since equation (1) solves Problem 4 for regular polygons, we are interested in irregular polygons with *some* rotational symmetry. That is, there must be some angle $\theta < 360°$ that we can rotate the polygon and the resulting shape is identical to the initial one ($\theta = 360°$ means there is no rotational symmetry). Figure 20 – Figure 23 show various polygons with their (minimal) rotational symmetries. Notice that the regular decagon also has rotational symmetries of 72° and 180° besides the 36°.

We are interested in the minimal rotational symmetry θ. In particular this means θ divides 360° evenly. That is, there is some integer m such that $m\,\theta = 360°$. Notice that the number of rotations m that restore the polygon to its original position is also the number of sides of the polygon that have the same length, since one side must be able to replace another without showing any difference. If m = n, the number of sides of the original polygon, then that polygon is regular—all its sides are the same length and span the same angle. If m < n, then m must divide n, so that there is an integer p such that n = m p, since the m rotations that move the m sides into identical positions also move the remaining sides into their identical positions (see Figure 24).

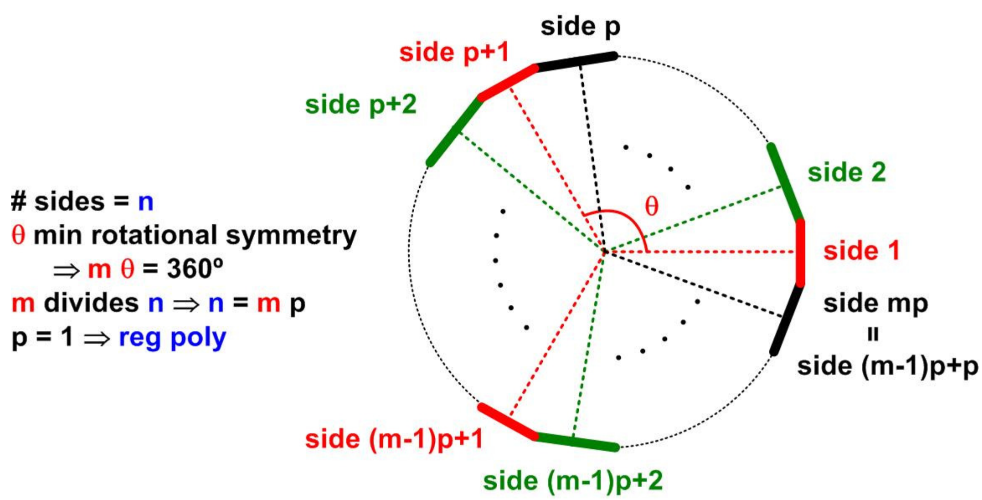

Figure 24 Rotational symmetry of irregular n-sided polygon

Now the key to the sum of perpendiculars being constant is that the corresponding unit vectors are separated by the same angle. So for each of the p sets of m sides of the polygon the corresponding unit vectors have this property. Therefore they each will sum to **0** and so the projections will cause a net 0 change in the lengths as before.

For example, consider the "diamond" 6-sided polygon in Figure 21 and the usual vector **v** moving the center point P to any point P' in the interior of the polygon with perpendiculars to the (extended) sides:

$$\mathbf{v}\cdot\mathbf{u_a} + \mathbf{v}\cdot\mathbf{u_b} + \mathbf{v}\cdot\mathbf{u_c} + \mathbf{v}\cdot\mathbf{u_d} + \mathbf{v}\cdot\mathbf{u_e} + \mathbf{v}\cdot\mathbf{u_f} = \mathbf{v}\cdot(\mathbf{u_a} + \mathbf{u_c} + \mathbf{u_e} + \mathbf{u_b} + \mathbf{u_d} + \mathbf{u_f}) = \mathbf{v}\cdot(\mathbf{0} + \mathbf{0}) = \mathbf{v}\cdot\mathbf{0} = 0$$

The difficulty comes when we try to compute the constant sum. We no longer have the simple equation (1). Now we have (in the case p = 2 for example)

$$\mathbf{a_1} + \mathbf{a_2} + \mathbf{a_3} + \mathbf{a_4} + \ldots + \mathbf{a_{2m-1}} + \mathbf{a_{2m=n}} = m\,\mathbf{a_1} + m\,\mathbf{a_2} = m\,\mathbf{S_1}/2 / \tan \theta_1/2 + m\,\mathbf{S_2}/2 / \tan \theta_2/2$$

where $\theta_1 + \theta_2 = \theta$, the minimal rotation symmetry. If the sides S_k and angles θ_k are unknown, we can't evaluate the constant sum.

Alternative Method for Problem 4

Applying a Polya idea to transform the problem to a limiting case, I imagined what would happen as the number of sides of the regular polygons increased, say to infinity, that is, a circle. Then the ideas of "adding" up lines from a point to the boundary of the circle is reminiscent of the old Cavalieri "indivisible" line arguments in the early years of the calculus. So that made me think of transforming these problems into "equivalent" area problems (eventually using "infinitesimals" that have non-zero area instead of the lines).

Convex Polygons With n Equal Sides (Regular Polygons)

We begin again with regular polygons. Figure 25 and Figure 26 show a regular hexagon and regular octagon subdivided into triangular areas with a common vertex at the random point P' inside the polygon. These triangles sum to the total area inside the polygon no matter where P' ends up.

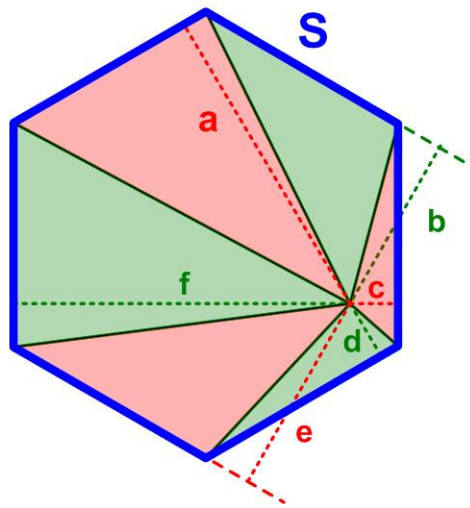

Figure 25 Regular hexagon with areas

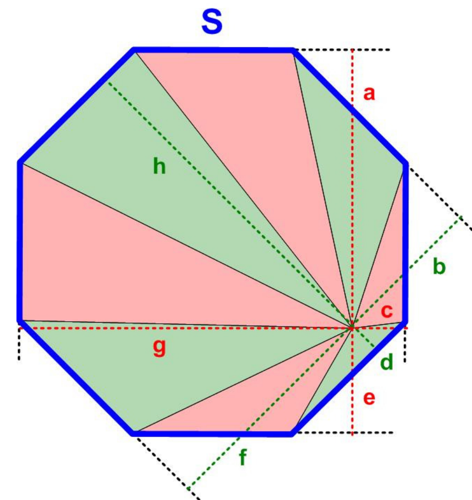

Figure 26 Regular octagon with areas

So for the hexagon we have

$$\tfrac{1}{2}\, a\, S + \tfrac{1}{2}\, b\, S + \tfrac{1}{2}\, c\, S + \tfrac{1}{2}\, d\, S + \tfrac{1}{2}\, e\, S + \tfrac{1}{2}\, f\, S$$

$$= \tfrac{1}{2}\, S\, [a + b + c + d + e + f] = \text{Area of hexagon}$$

Therefore, the sum of the perpendiculars must be a constant, as before. From the central position for P we can compute the area of one triangle using equation (1) for the altitude

$$\mathbf{a} = \mathbf{S}/2 \,/\, \tan 180°/6 = \mathbf{S}/2 \,/\, \tan 30° = \mathbf{S}/2 \,/\, 1/\sqrt{3} = \sqrt{3}/2\, S$$

$$\text{Area of triangle} = \tfrac{1}{2}\, S\, (\sqrt{3}/2\, S) = \sqrt{3}/4\, S^2$$

$$\text{Area of hexagon} = 6 \text{ Area of triangle} = 3\sqrt{3}/2\, S^2$$

Therefore,

$$a + b + c + d + e + f = \text{Area of hexagon}/\, S/2 = 3\sqrt{3}\, S$$

as before. Clearly this approach works for any regular polygon.

Notice these areas are of the form

$$\tfrac{1}{2}\frac{L}{n}\sum_{k=1}^{n} altitude_k = \tfrac{1}{2} L \sum_{k=1}^{n} \frac{altitude_k}{n} \qquad (2)$$

where L is the perimeter of the polygon. That is, the area of the regular polygon becomes L/2 times the average of the altitudes. Therefore, for any regular polygon of n sides (actually any convex polygon whose n sides are equal), all we have to do is find its area, divide by half the perimeter, and multiply by n to get the constant sum of the altitudes from any point P inside the polygon. In other words, equation (2) is equivalent to the

Claim. *For any convex polygon of n equal sides and for any point in the interior of the polygon, the sum of the (perpendicular) distances from that point to each of the (possibly extended) sides of the polygon is a constant, and that constant is 2nA / L, where A is the area and L is the perimeter of the polygon.*

Viviani Theorem

Returning to the beginning, we can apply this Claim to get a geometric proof that the sum of perpendiculars from a point P in an equilateral triangle is a constant, since we already know the area of the triangle. (This is in fact the Viviani Theorem and proof. It is limited to equilateral triangles.) Decompose the equilateral triangle into three triangles with bases the edges of the equilateral triangle and vertices coinciding with P (Figure 27). Then

$$A = \tfrac{1}{2} S \sqrt{3}/2\, S = \tfrac{1}{2} a S + \tfrac{1}{2} b S + \tfrac{1}{2} c S = \tfrac{1}{2} S (a + b + c)$$

$$\Rightarrow a + b + c = \sqrt{3}/2\, S \quad \text{as before.}$$

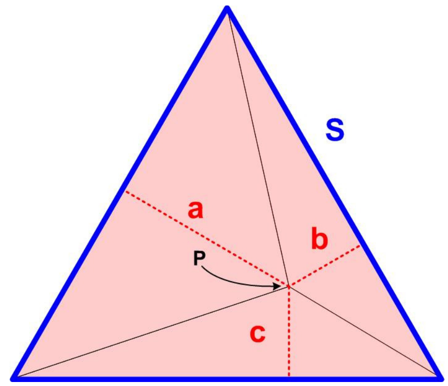

Figure 27 Viviani Theorem

Convex Polygons with Rotational Symmetries

Now we consider the general problem of irregular polygons with rotational symmetries. The discussion before about subdividing the sides with common length and rotational symmetry θ still hold. That is, θ must divide 360° evenly so that m θ = 360°. Again if n is the number of sides to the polygon, then m divides n, so that n = m p for some integer p (see Figure 24 and examples Figure 28 and Figure 29).

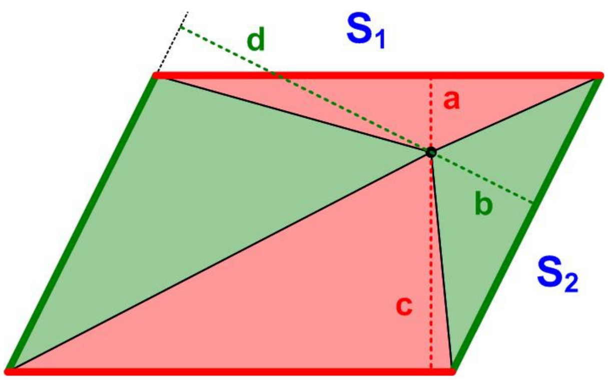

Figure 28 Parallelogram with areas (p = 2)

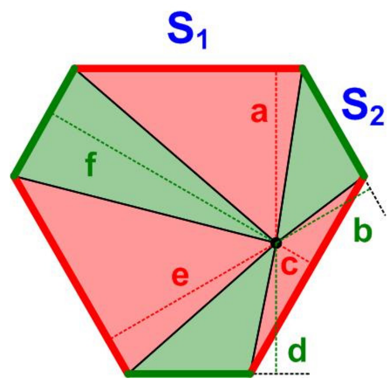

Figure 29 Diamond with areas (p = 2)

For example, consider the diamond 6-sided polygon in Figure 29. The areas become

$$\tfrac{1}{2} a S_1 + \tfrac{1}{2} c S_1 + \tfrac{1}{2} e S_1 + \tfrac{1}{2} b S_2 + \tfrac{1}{2} d S_2 + \tfrac{1}{2} f S_2$$

$$= (a + c + e) \tfrac{1}{2} S_1 + (b + d + f) \tfrac{1}{2} S_2$$

$$= \text{Area of Diamond} \qquad (3)$$

Looks like there may be a problem with this approach. The relationship (3) is of the form

$$Ax + By = C$$

where $A = \tfrac{1}{2} S_1$, $B = \tfrac{1}{2} S_2$, C = Area of Diamond, $x = (a + c + e)$, and $y = (b + d + f)$. The locus of points (x, y) is a straight line, that is, x and y can vary in a dependent way, namely, $y = -A/B\, x + C/B$.

There is a constraint: x and y must be positive. But that amounts to saying neither the sum of green areas nor the sum of red areas can be the entire area, which is true. So it is not immediately evident to me what condition would render x and y each constant.

Aha! We need to reduce it to a previous problem we have solved, namely, the regular polygons. Figure 30 indicates the way out. For any of the set of m identical sides, we can extend them to an m-sided regular polygon. So in the Diamond example, we can extend the S_2 sides to an equilateral triangle with sides of length S_2'. Then

$$(b + d + f)\ \tfrac{1}{2}\ S_2' = \text{Area of the triangle T}$$

Therefore,

$$(b + d + f)\ \tfrac{1}{2}\ S_2 = T\ S_2 / S_2' = \text{constant}$$

$$\Rightarrow b + d + f = \text{constant}$$

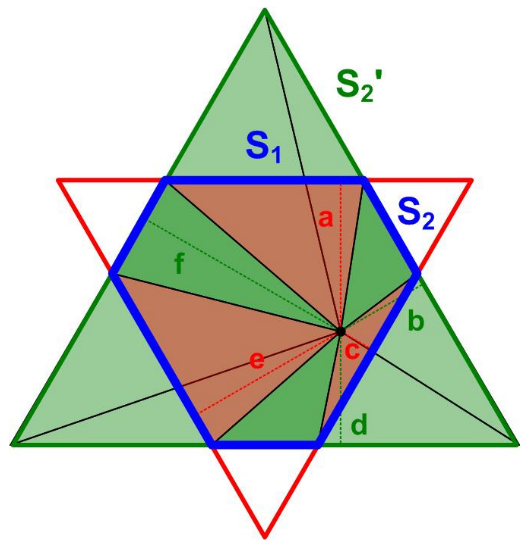

Figure 30 Regular 3-sided polygon extension (triangle)

(Of course, we already knew from the triangle that b + d + f = constant. But the subsequent computations make clear how the original setting was constant.) Repeating this for the other sides yields the final answer.

Convex Figures with Straight or Curved Perimeters

Now we return to the idea mentioned initially regarding the area approach: taking limits of "infinitesimals".

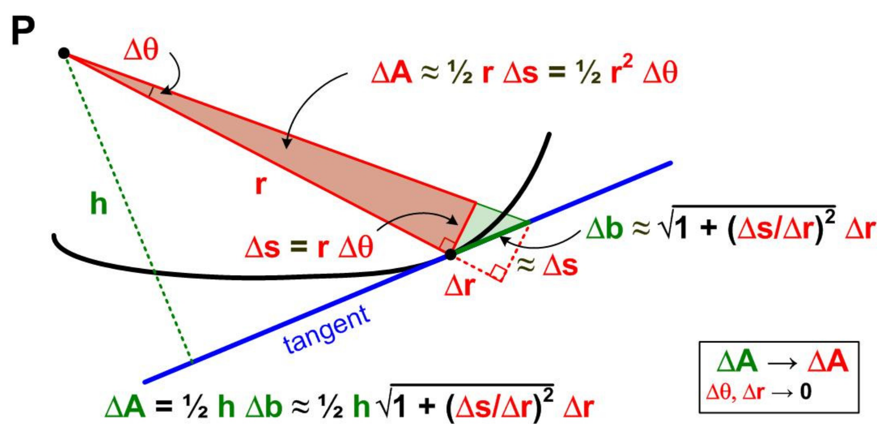

Figure 31

The average altitude idea even works for a circle or any convex figure (the line between any two points in the figure also lies in the figure). First, notice that equation (2) can be written

$$\tfrac{1}{2}\frac{L}{n}\sum_{k=1}^{n} altitude_k = \sum_{k=1}^{n}\tfrac{1}{2}\Delta b_k h_k = \sum_{k=1}^{n}\Delta A_k = A \qquad (4)$$

where the sides of the regular polynomial become possibly arbitrary lengths Δb_k corresponding to arbitrary segments of the sides of non-regular polynomials and h_k represent the altitudes from the fixed point P to the (extended) segment Δb_k.

There is another way of representing an infinitesimal area ΔA_k that involves the distance along the slant line from the fixed point P to an arbitrary point on the boundary of the convex figure. This sort of corresponds to the slant line cases of the polynomials considered in the previous problems. Figure 31 shows a slant line of length r from P to a point on the perimeter of the convex figure. The blue line represents the tangent line to the perimeter at the point. A small wedge of area can be constructed based on a small change in angle $\Delta\theta$ at P. One way is shown in red bounded by the radial length r on both sides and the arc length $\Delta s = r\Delta\theta$ at the base. The arc length Δs meets the radial length r at a right angle. Then $\Delta A = \frac{1}{2} r\, \Delta s = \frac{1}{2} r^2\, \Delta\theta$. As $\Delta\theta$ shrinks to zero, the infinitesimal area ΔA looks more and more like the slant length r, so adding up all the infinitesimal areas around the convex figure looks like adding up all the slant lines.

This infinitesimal area is also approximated by another type of wedge similar to the triangular areas we computed for the regular polygons. Using the tangent line to the point on the perimeter, we form a small base Δb along the tangent line determined by the wedge of angle $\Delta\theta$, namely $\Delta b \approx \sqrt{1+(\frac{\Delta s}{\Delta r})^2}\,\Delta r$ (Figure 31). Then $\Delta A = \frac{1}{2} h\, \Delta b = \frac{1}{2} h \sqrt{1+(\frac{\Delta s}{\Delta r})^2}\,\Delta r$ where h is the altitude from P to the tangent line. Now as $\Delta\theta, \Delta r \to 0$, $\Delta A \to \Delta A$. So either infinitesimal should give the same result in the limit.

In terms of calculus, we have for the area A inside the figure determined by lines from a point P, first using the scheme of equation (4),

$$A = \lim_{\substack{\Delta\theta\to 0\\ n\to\infty}} \sum_{k=1}^{n} \Delta A_k = \lim_{\substack{\Delta\theta\to 0\\ n\to\infty}} \sum_{k=1}^{n} \tfrac{1}{2} h_k \Delta b_k = \lim_{\substack{\Delta\theta\to 0\\ n\to\infty}} \sum_{k=1}^{n} \tfrac{1}{2} h_k \sqrt{1+(\tfrac{\Delta s_k}{\Delta r_k})^2}\,\Delta r_k = \int \tfrac{1}{2} h(r) \sqrt{1+(\tfrac{ds}{dr})^2}\, dr \qquad (5)$$

$$A = \lim_{\substack{\Delta\theta\to 0\\ n\to\infty}} \sum_{k=1}^{n} \Delta A_k = \lim_{\substack{\Delta\theta\to 0\\ n\to\infty}} \sum_{k=1}^{n} \tfrac{1}{2} r_k^{\,2} \Delta\theta_k = \int \tfrac{1}{2} r(\theta)^2\, d\theta \qquad (6)$$

From equation (5) we can define the "average altitude" $\bar{h}$ via

$$A = \int \tfrac{1}{2} h(r) \sqrt{1+(\tfrac{ds}{dr})^2}\, dr = \bar{h} \int \tfrac{1}{2} \sqrt{1+(\tfrac{ds}{dr})^2}\, dr = \bar{h}\, \tfrac{1}{2} L$$

where again L is the length of the perimeter of the convex figure. So for a circle of radius r, we have the average "average altitude" $\bar{h}$ = Area / ½ Perimeter = $\pi r^2 / \pi r = r$, as expected.