Nahin Triangle Problem

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I have been reading with interest Paul J. Nahin's latest book *Number-Crunching* [1]. On page 29 and following Nahin presents a problem that he will solve with the Monte Carlo sampling approach. Here is his statement of the problem ([1], p.29):

> To start, imagine an equilateral triangle with side lengths 2, as shown in [Figure 1]. If we pick a point "at random" from the interior of the triangle, what is the probability that the point is no more distant than $d = \sqrt{2}$ from each of the triangle's three vertices? The shaded region in the figure is where all such points are located. There is nothing special about the $\sqrt{2}$ other than it will make some of the theoretical calculations we'll do, to check the Monte Carlo computer code, particularly simple to perform. We could, however, solve the problem for different values of d. The exact theoretical answer (for $d = \sqrt{2}$) is
>
> $$\frac{\pi}{2\sqrt{3}} + 1 - \sqrt{3} = 0.1748488... \qquad (1)$$
>
> The theoretical calculation of [(1)] requires mostly only high school geometry, plus one step that I think requires a simple freshman calculus computation.

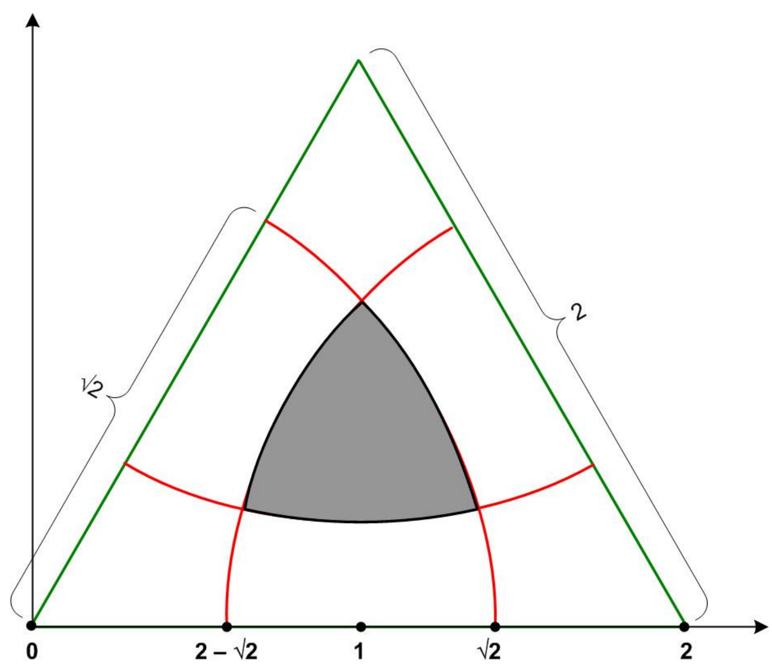

Figure 1 The points in the shaded region are all the points within $d = \sqrt{2}$ of all three vertices of the equilateral triangle

I thought I would try to find the analytic solution in equation (1). I believe I succeeded without calculus, unless the formula for the area of a sector is considered calculus (see Figure 2). My solution follows on the next page using only geometry and the area of triangles and sectors. It differs from the one provided by Nahin in the back of his book which does use calculus at one point.

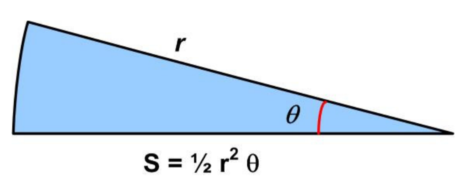

Figure 2 Area of Sector

Solution. So we are interested in finding the ratio of the area of the shaded "triangle" in Figure 1 and the equilateral triangle.

We partition the equilateral triangle into 7 areas created by the intersecting circular arcs of radius $\sqrt{2}$, where area A_3 is the desired shaded triangle (see Figure 3). By the nature of all the symmetries there are two sets of 3 areas each where the areas in each set are all the same and so have been labeled with the same subscript. If we let T be the area of the equilateral triangle, then

$$T = 3A_1 + 3A_2 + A_3 \tag{2}$$

Similarly the partition of the triangle partitions each sector (Figure 4). If S represents the area of such a sector, then

$$S = A_1 + 2A_2 + A_3 \tag{3}$$

and so

$$T = 2A_1 + A_2 + S \tag{4}$$

From equation (2) we get

$$A_3 = T - 3A_1 - 3A_2 \tag{5}$$

Using equation (4) to eliminate A_2 in equation (5) yields

$$A_3 = 3S - 2T + 3A_1 \tag{6}$$

or, evaluating the areas of the sector and equilateral triangle,

$$A_3 = \pi - 2\sqrt{3} + 3A_1 \tag{7}$$

where we recall that the angles in an equilateral triangle are each 60° or $\pi/3$ radians and the altitude of this equilateral triangle is $\sqrt{3}$. It remains, then, to compute the value of the area A_1.

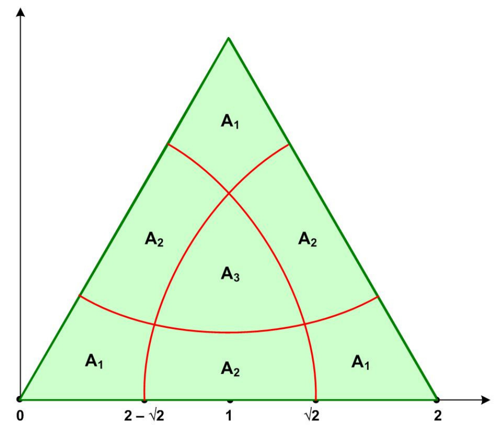

Figure 3 Partitioned equilateral triangle

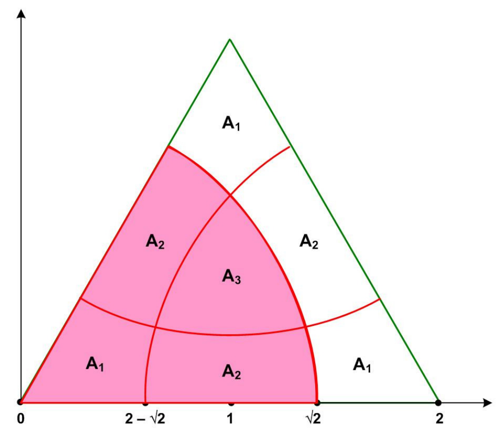

Figure 4 Partitioned sector

Area A_1

As shown in Figure 5, drop the perpendicular bisector (dashed line) on the side of the equilateral triangle opposite the origin. Every point along this line is equidistant from the two vertices defining the side of the triangle. (If we join the point by lines to each vertex, we define two congruent right triangles (s.a.s), and so the lines (hypotenuses) must be the same length.) This equidistant property means the perpendicular bisector intersects the third vertex in the equilateral triangle at the origin. It also means the point on the bisector that is $\sqrt{2}$ from each of the vertices opposite the origin is the intersection of the two circular arcs centered at

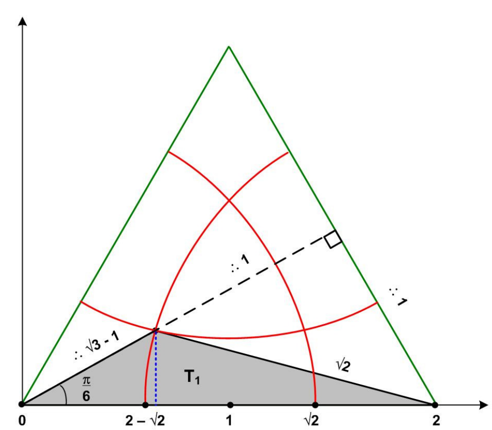

Figure 5 Triangle T_1

these vertices. This $\sqrt{2}$ distance represents the hypotenuse of a right triangle with one side of length 1 (since the perpendicular bisects the side of length 2 in half). Therefore the other side of the triangle is also of length 1. Since the perpendicular bisector is also the altitude of the equilateral triangle, it is of length $\sqrt{3}$, which means the balance of the length of the altitude from the point of intersection of the arcs to the vertex at the origin is $\sqrt{3} - 1$.

Consider the shaded triangle T_1 in Figure 5 with base the same as the equilateral triangle and top vertex at the intersection of the two circular arcs. To compute the area T_1 we need the altitude (blue dashed line). The argument about the congruent right triangles using the perpendicular bisector means the two right triangles determined by the bisector and vertices of the equilateral triangle are congruent. This implies the perpendicular bisector also bisects the vertex angle of the equilateral triangle and so its value is 30° or $\pi/6$ radians. This implies the altitude of T_1 is half the hypotenuse of the left hand right triangle, namely $(\sqrt{3} - 1)/2$, so that the area of T_1 is

$$T_1 = \tfrac{1}{2}\ 2\,(\sqrt{3} - 1)/2 = (\sqrt{3} - 1)/2 \qquad (8)$$

From Figure 6 we see that the triangle T_1 can be partitioned into half of area A_1 and a sector S_1. so that

$$T_1 = \tfrac{1}{2}\ A_1 + S_1 \qquad (9)$$

From Figure 7 we see that the right triangle with equal sides implies the angle at its vertex is 45° or $\pi/4$ radians. This means the angle defining the sector S_1 is $\pi/3 - \pi/4 = \pi/12$. Therefore,

$$S_1 = \tfrac{1}{2}\,(\sqrt{2})^2\,\pi/12 = \pi/12 \qquad (10)$$

So from (8), (9), and (10), we get

$$\boxed{A_1 = \sqrt{3} - 1 - \pi/6} \qquad (11)$$

Plugging this value into equation (7) yields

$$A_3 = \pi/2 + \sqrt{3} - 3 \qquad (12)$$

Thus the probability of a point randomly falling inside the equilateral triangle being less than $\sqrt{2}$ from each of the three vertices is A_3/T or (since $T = \tfrac{1}{2}\,2\,\sqrt{3} = \sqrt{3}$)

$$\boxed{\frac{\pi}{2\sqrt{3}} + 1 - \sqrt{3}} \qquad \text{QED}$$

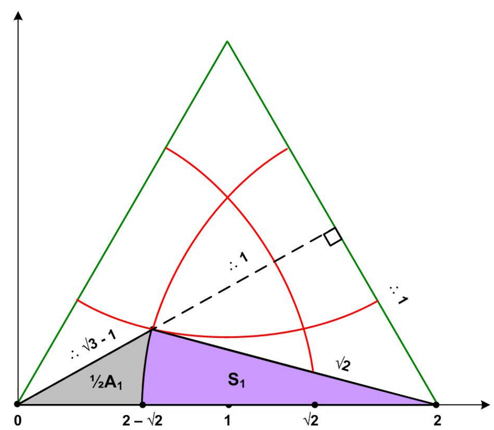

Figure 6 Components of T_1 triangle

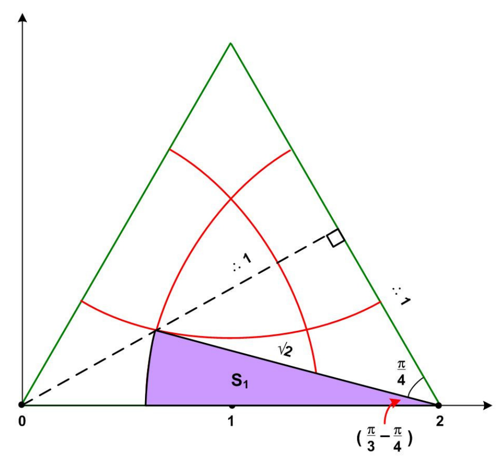

Figure 7 Area of sector S_1

References.

[1] Nahin, Paul J., *Number-Crunching: Taming Unruly Computational Problems from Mathematical Physics to Science Fiction*, Princeton University Press, 2011