Meditation on "Is" in Mathematics Part I – Zeno's Paradox

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The following is a meditation on the nature of mathematics as I see it. I have been thinking about this for some time, and my thoughts were again stimulated by a March 2014 article I read in *Slate* attempting to popularize mathematical concepts associated with Zeno's Paradox. In Part I, I will first present the article, heavily annotated with my critique. Then in Part II I will try to explain in more depth the admittedly philosophical concepts I am trying to get at.

What Is the Answer to Zeno's Paradox?

Why Achilles actually can catch a tortoise in a race.

By Brian Palmer, 5 March 2014

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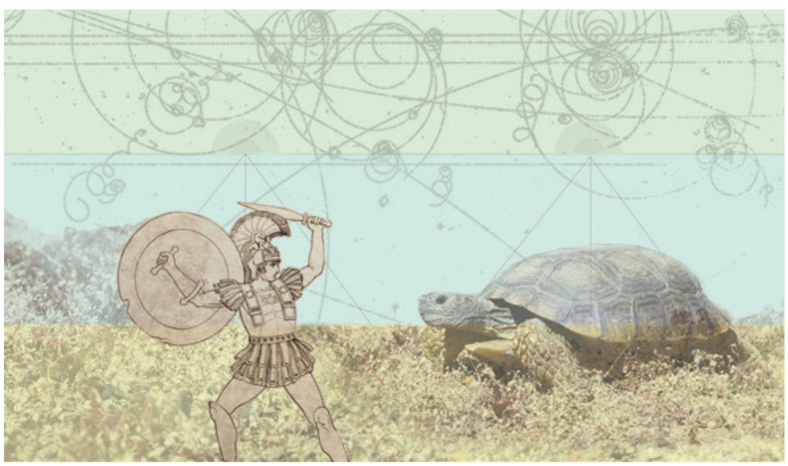
(http://www.slate.com/articles/health_and_science/science/2014/03/zeno_s_paradox_how_to_explain_the_solution_to_achilles_and_the_tortoise.html, retrieved 2/5/15)



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The Greek philosopher Zeno[1] wrote a book of paradoxes nearly 2,500 years ago. "Achilles and the Tortoise" is the easiest to understand, but it's devilishly difficult to explain away. For those who haven't already learned it, here are the basics of Zeno's logic puzzle, as we understand it after generations of retelling:

> Achilles, the fleet-footed hero of the Trojan War, is engaged in a race with a lowly tortoise, which has been granted a head start. Achilles' task initially seems easy, but he has a problem. Before he can overtake the tortoise, he must first catch up with it. While Achilles is covering the gap between himself and the tortoise that existed at the start of the race, however, the tortoise creates a new gap. The new gap is smaller than the first, but it is still a finite distance that Achilles must cover to catch up with the animal. Achilles then races across

[1] JOS: 490 – 430 BC.

> the new gap. To Achilles' frustration, while he was scampering across the second gap, the tortoise was establishing a third. The upshot is that Achilles can never overtake the tortoise. No matter how quickly Achilles closes each gap, the slow-but-steady tortoise will always open new, smaller ones and remain just ahead of the Greek hero.

It's tempting to dismiss Zeno's argument as sophistry, but that reaction is based on either laziness or fear. Laziness, because thinking about the paradox gives the feeling that you're perpetually on the verge of solving it without ever doing so—the same feeling that Achilles would have about catching the tortoise. Fear, because being outwitted by a man who died before humans conceived of the number zero[2] delivers a significant blow to one's self-image. But what if your 11-year-old[3] daughter asked you to explain why Zeno is wrong?[4] Would you just tell her that Achilles is faster than a tortoise, and change the subject? That would be pretty weak. Zeno assumes that Achilles is running faster than the tortoise, which is why the gaps are forever getting smaller. But it doesn't answer the question.

Let's see if we can do better. I consulted a number of professors of philosophy and mathematics. Most of them insisted you could write a book on this (and some of them have[5]), but I condensed the arguments and broke them into three parts.

Step 1: Yes, it's a trick. But what kind of trick?

Zeno devised this paradox to support the argument that change and motion weren't real. Nick Huggett, a philosopher of physics at the University of Illinois at Chicago, says that Zeno's point was "Sure it's crazy to deny motion, but to accept it is worse."

The paradox reveals a mismatch between the way we think about the world and the way the world actually is.[6] (This seems obvious, but it's hard to grapple with the paradox if you don't articulate this point.) Joseph Mazur, a professor emeritus of mathematics at Marlboro College and author of the forthcoming book *Enlightening Symbols*,[7] describes the paradox as "a trick in making you think about space, time, and motion the wrong way."

The challenge then becomes how to identify what precisely is wrong with our thinking.[8] Motion is possible, of course, and a fast human runner can beat a tortoise in a race. The problem has

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2 http://www.scientificamerican.com/article/history-of-zero/

3 http://www.slate.com/articles/health_and_science/science/2014/02/flame_challenge_2014_what_is_color_alan_alda_has_kids_judge_scientists.html

4 JOS: "Wrong" is a heavy word here. This assumes Zeno made a mistake, which is not necessarily the case. It is possible that correct reasoning could lead to a contradiction, in which case somewhere along the line the current assumptions were inconsistent. We are optimistic that we can correct the situation and so use the word "paradox." New ideas are needed to *resolve* the paradox (make it go away). As we shall see, where we got off the track is assuming infinite processes behave the same as finite operations.

5 http://www.amazon.com/dp/0452289173/?tag=slatmaga-20

6 JOS: OK. This is the right way to view it.

7 http://www.amazon.com/dp/0691154635/?tag=slatmaga-20

8 JOS: Again, I would say "deficient" and not "wrong" regarding our thinking. "Wrong" suggests there is a pre-existing "right" that we just have not seen in the current state of affairs. We do have an apparent contradiction: physically we know someone running faster than someone else will certainly pass them. The problem here is that a mathematical representation of the situation does not seem to agree. "Deficient" suggests we are lacking something that we need to add in order to resolve the problem (since we are always optimistic that math can be changed to keep reflecting physical reality). This is the more fundamental process in mathematical and physical advancements.

something to do with our conception of infinity.[9]

Step 2: There's more than one kind of infinity.

Achilles' task seems impossible because he "would have to do an infinite number of 'things' in a finite amount of time," notes Mazur, referring to the number of gaps the hero has to close. But not all infinities are created the same.[10]

There are divergent series and convergent series.[11] The most obvious divergent series is 1 + 2 + 3 + 4 ... There's no answer to that equation.[12] Or, more precisely, the answer is "infinity."[13] If Achilles had to cover these sorts of distances over the course of the race—in other words, if the tortoise were making progressively larger gaps rather than smaller ones—Achilles would never catch the tortoise.

Now consider the series 1/2 + 1/4 + 1/8 + 1/16 ... Although the numbers go on forever, the series converges, and the solution is 1.[14] As long as Achilles is making the gaps smaller at a sufficiently fast rate, so that their distances look more or less like this equation, he will complete the series in a measurable amount of time and catch the tortoise.

There's a little wrinkle here.[15] It will muddy the waters, but intellectual honesty compels me to tell you that there is a scenario in which Achilles doesn't catch the tortoise, even though he's faster. "It is mathematically possible for a faster thing to pursue a slower thing forever and still never catch

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9 JOS: Actually the problem was that the *Greeks* did not *have* a viable concept of infinity in 500 BC, nor did the rest of the western world until after the Middle Ages some 2000 years later. And most people today, untutored in mathematics, are at the same level of understanding about this as the Greeks in 500 BC.

10 JOS: True, but misleading in this case. The issue here is not about countable vs. uncountable infinities; it is about what will be referred to as convergent vs. divergent infinite series. Or more to the point, how can an *infinite* sum of numbers (as indicated in the Achilles and Tortoise problem) be interpreted to "add" up to a finite number that would be consistent with our actual physical experience (and the rules of arithmetic we already have)?

11 JOS: These terms are introduced in a stand-alone way, rather than as a consequence of defining the notion of an infinite sum. This is a poor pedagogical approach.

12 JOS: Again, sloppy wording. First there is no equation, and the word "answer" implies there is a question without actually saying what it is. Implicitly, Palmer is trying to ask when does an infinite sum of positive quantities yield a finite value. Our intuition would suggest, "never." So we first need to *define* what such a "finite" value to an infinite sum would *mean*, since our intuition says any infinite sum should grow without bound (which is the heart of the Zeno paradox).

13 JOS: Again, this is poor wording—this isn't precise at all. Maybe this is the "other" infinity Palmer was suggesting by his title. Here Palmer means "infinity" in the sense of growing without bound, which he should state instead of using the word infinity. We are already thinking of infinity in terms of an "infinite" number of positive quantities which we are summing up, and that is already a source of confusion.

14 JOS: This is at the heart of the business and is presented in an off-hand way without explanation. The impression is given that this infinite sum *is* 1, when our intuition says *no infinite* sum can have a finite value. Where did 1 come from? What does it mean to say the sum *is* 1? How do you know? The crux here is that over the centuries mathematicians came to extend their notions of mathematics from the finite to the realm of the infinite in a *consistent* and *plausible* way. What does that mean? It means there is a way to *extend* our notion of a finite sum of positive numbers (even positive or negative numbers) to an infinite sum, by ultimately introducing the notion of *limit*. Where the extension can be defined, the infinite sum or infinite series is said to *converge*. Where we cannot meaningfully extend the notion of sum, the infinite series is said to *diverge*. We can show (though not here) the series 1/2 + 1/4 + 1/8 + 1/16 + … does not grow without bound, and in fact grows closer and closer to 1 as we add more terms. So it would be *reasonable* to *define* the infinite sum to be 1 in this case.

15 JOS: It is not pedagogically sound to discuss subtleties to a concept that has just been introduced and is only vaguely understood. In fact, it tends to make the concept even harder to understand.

it," notes Benjamin Allen, author of the forthcoming book *Halfway to Zero*, "so long as both the faster thing and the slower thing both keep slowing down in the right way."

The secret again lies in convergent and divergent series. For example, the series 1/2 + 1/3 + 1/4 + 1/5 ... looks convergent, but is actually divergent.[16] If Achilles runs the first part of the race at 1/2 mph, and the tortoise at 1/3 mph, then they slow to 1/3 mph and 1/4 mph, and so on, the tortoise will always remain ahead. But don't tell your 11-year-old about this. It will be our little secret.

Step 3: It's more than a theory.

If your 11-year-old is contrarian by nature, she will now ask a cutting question: How do we know that 1/2 + 1/4 + 1/8 + 1/16 ... adds up to 1? No one has ever completed, or could complete, the series, because it has no end. The conclusion that an infinite series can converge to a finite number is, in a sense,[17] a theory, devised and perfected by people like Isaac Newton and Augustin-Louis Cauchy, who developed an easily applied mathematical formula[18] to determine whether an infinite series converges or diverges. But thinking of it as only a theory is overly reductive.[19]

"It's easy to say that a series of times adds to [a finite number]," says Huggett, "but until you can explain in general—in a consistent way—what it is to add any series of infinite numbers, then it's just words. Cauchy gave us the answer."[20]

The convergence of infinite series explains[21] countless things we observe in the world. Not just the fact that a fast runner can overtake a tortoise in a race, either. Any distance, time, or force that exists in the world can be broken into an infinite number of pieces—just like the distance that Achilles has to cover—but centuries of physics and engineering work have proved that they can be treated as finite.

That answer might not fully satisfy ancient Greek philosophers, many of whom felt that their

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[16] JOS: That is, in this case the series grows without bound, and so does not allow a finite value to be defined as its sum in any meaningful way. Proving this is not immediately obvious and may only cause confusion at this stage. Clearly what Palmer is trying to get at is that he has been presenting cases of infinite sums where the sum cannot be defined when the nth term does not go to zero, and can be defined in a case where the nth term does go to zero. He just wants to explain that the nth term going to zero is a necessary but not sufficient condition. That is, some infinite sums still do not converge even when their individual terms shrink to zero. This would be an acceptable topic if he were explaining the whole notion of convergent and divergent series in more detail. Mentioning it now tends to undermine the intuition that is *basically* true, namely, the only way to succeed in assigning a number to an infinite sum is if its successive individual terms are shrinking fast enough (to zero) to become essentially "insignificant," that is, the error in stopping the addition after a certain *finite* number of terms is negligible.

[17] JOS: Again, this is misleading. It belittles the idea that we are making something up when we assign a finite value to an infinite sum, but that in fact is *exactly* what we are doing. The infinite series has no *inherent, actual* sum before we invented one. That is, all our previous notions involved the addition of a *finite* number of things, so this *infinite* situation is uncharted waters.

[18] http://www.math.ubc.ca/~cass/courses/m220-00/cauchy.pdf

[19] JOS: Again, belittling and misleading, and ignoring the true state of affairs.

[20] JOS: A novice might misconstrue this sentence, thinking "explain" and "answer" reveal what was already there, when again mathematically these words mean "define." From Bressoud's article *True Grit in Real Analysis* (1998, 2014) "… mathematicians are not using definitions as they are usually encountered, as descriptions of entities that already exist. For mathematicians, definitions are prescriptive."

[21] JOS: Rather "applies to", since in general infinite series do not explain phenomena, though some might argue the expansion of the quantum wave function into an infinite series of orthogonal pure state functions is an explanation, or the expansion of a sound wave into a series of harmonic sinusoidal terms explains how it is made up of a superposition of frequencies.

logic was more powerful than observed reality.[22] But the way mathematicians and philosophers have answered Zeno’s challenge, using observation to reverse-engineer a durable theory,[23] is a testament to the role that research and experimentation play in advancing understanding. It should give pause to anyone who questions the importance of research in any field.[24]

My comments on this article may appear too nit-picking. Much of Palmer’s writing could be construed to convey the same ideas I am getting at. The problem as I see it is that it can also be quite misconstrued, and so lead to real confusion and lack of understanding. Hopefully the rest of my remarks in Part II will explain the distinctions I was trying to make in the annotations and reveal more about the true nature of mathematics (or at least my take on it).

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[22] JOS: Again this misses the mark. The Greeks were not favoring philosophy over reality. They had successfully built a powerful and logically consistent edifice of basic mathematics and could not figure out how to extend it to the realm of the infinite. They were aware that their tools could not capture the problem of motion (and other things based on infinite arguments), thus Zeno’s Paradoxes, so they abandoned the effort and stuck with what they could handle in their current system.

[23] JOS: This is strong poetic license. There was no reverse engineering of reality! It actually took the mental adjustment of allowing the infinite to become acceptable, via an infinite God from the Middle Ages, that led in turn to the wobbly notion of infinitesimals and eventually the calculus of Newton and Leibniz in the 17th century. In the 18th century Cauchy put the ideas on a more substantial logical foundation with his precise definitions of limits.

[24] JOS: Finally, a truism, for it was through “research” that the real ideas of grappling with the infinite were achieved.