Hyperboloid as Ruled Surface

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When our daughter-in-law made wheat shocks as center-pieces for her fall-themed wedding reception, I naturally could not help pointing out the age-old observation that they represented a hyperboloid of one sheet (Figure 1). This was naturally greeted with the usual groans, but the thought stayed with me as I realized I had never proved this mathematically to myself.

Figure 1 Wheat shock center-piece

The way the wheat shocks were made follows the time-honored way of making a string or wire model of the hyperboloid, as shown in Figure 2 ([1]). Namely, strings or wires are stretched vertically between two flat disks. The disks are twisted in opposite directions, yielding the subsequent images with increased twisting. A red line superimposed over one of the strings shows the amount of deviation from the vertical the string makes as the disks are twisted. A horizontal cross-section of the surface represented by the strings is a circle, indicated in green in the figures.

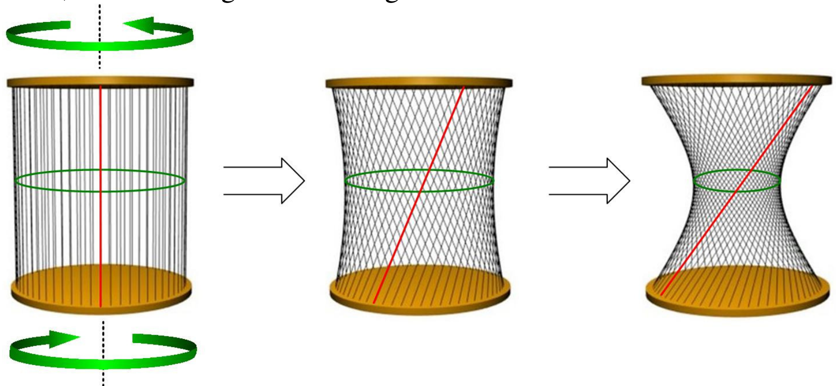

Figure 2 Twisting a cylinder of strings into a hyperboloid

Two mathematical concepts are involved here: a ruled surface and a hyperboloid of one sheet.

Ruled Surface

The intuitive notion of a ruled surface involves a curve and a line moving along the curve sweeping out a two-dimensional ribbon-like shape as it bobs and twists along the curve. Figure 3 is an illustration of a ruled surface where the curve is shown in green and the line (segment) is shown in red.

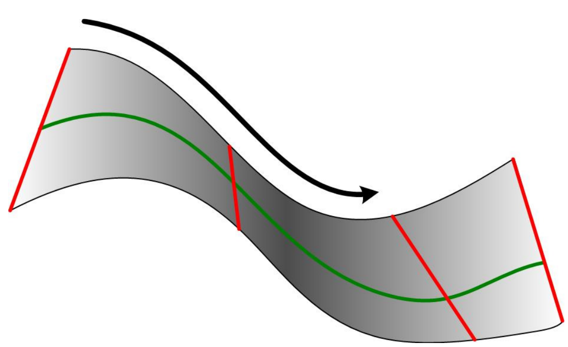

Figure 3 A generic ruled surface

As can be seen from Figure 2, a cylinder is an example of a ruled surface where a vertical line moves around a closed curve (circle in this case) keeping the same angle with respect to the curve and not twisting as it moves. The end picture in Figure 2 suggests that the surface we are interested in is also a ruled surface, and that in fact it is a hyperboloid of one sheet. The point of this note is to prove this mathematically.

Hyperboloid of One Sheet

But first we need to define what a hyperboloid of one sheet is. It is the locus of points (x, y, z) in 3-dimensional space satisfying the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad (1)$$

where a, b, and c are constants (see Figure 4). Notice that the (green) points on the hyperboloid in the xy-plane (where z = 0) satisfy the equation for an ellipse:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

and likewise any horizontal slice parallel to the xy-plane intersects the hyperboloid in an ellipse with proportionately larger semimajor and semiminor axes (a and b, resp.). Similarly, the (red) points on the hyperboloid in the xz-plane (where y = 0) satisfy the equation for a hyperbola

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1$$

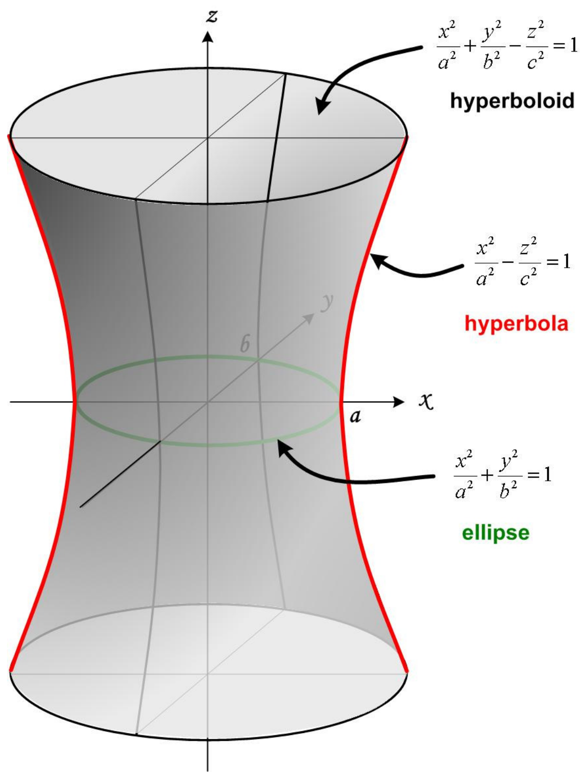

Figure 4 Hyperboloid of one sheet

Furthermore, we will assume the ellipse is a circle, that is, a = b. This more closely reflects the situation shown in Figure 2 and the wheat shock. Notice this means the hyperboloid is obtained by rotating the hyperbola (red curve) around the z-axis so that it is a surface of revolution. To simplify matters further we shall assume the radius of the circle is one and that c = 1 as well. So now we are considering the hyperboloid

$$x^2+y^2-z^2=1 \qquad (2)$$

and wish to show it is a ruled surface. Actually we shall proceed from the opposite direction by considering a ruled surface swept out by a line tilted to a 45° angle and moving around a circle of radius 1 (as modeled in Figure 2). We shall show it defines the hyperboloid given in equation (2).

Solution

Figure 5 shows an arbitrary point (x, y, z) on the ruled surface. It also shows the (red) generating line that passes through it from the (green) generating curve (circle). We wish to show (x, y, z) satisfies equation (2). Drop a perpendicular from the point down to the xy-plane. The length is z. Draw the line from the origin out to the foot of the perpendicular at (x, y). The length of this line is designated r and satisfies

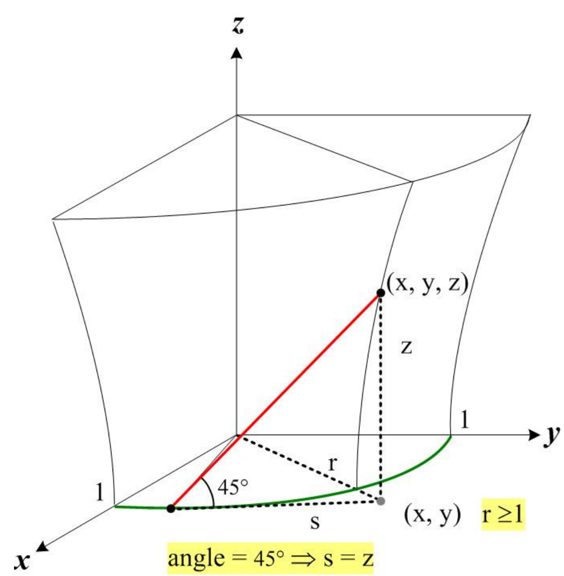

Figure 5 3D view

$$r^2 = x^2 + y^2$$

Extend the tangent to the circle at the point of intersection of the generating line with the generating circle to the foot of the perpendicular at (x, y) (see also Figure 6). The length of this line is designated s. From Figure 5 we can see that because the generating line makes an angle of 45° with the xy-plane we have $s = z$. Since a tangent to a circle is perpendicular to the radius at the point of tangency, we also have

$$s^2 = r^2 - 1 = x^2 + y^2 - 1$$

Hence,

$$z^2 = s^2 = x^2 + y^2 - 1$$

which is the same as equation (2), and we are done.

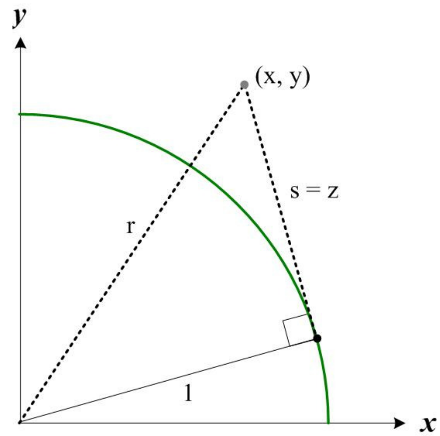

Figure 6 2D view

If we removed the restriction that the circle be of radius 1 and let it be any radius, and if we allowed the generating line to be tilted to any other angle than 45°, we would still get a hyperboloid of one sheet. This time it would be of the form

$$x^2 + y^2 - Az^2 = B$$

where A and B are constants (see [2], p.88). Investigating whether we would still get a hyperboloid if the generating curve were a general ellipse rather than a circle is a bit more complicated and I have not explored that yet.

I want to emphasize that this result is remarkable in a couple of ways. It is evident from Figure 2 that the resulting surface from the twisting is curved in some way, but it is not obvious that it should be a hyperboloid. The vertical cross-sections could be some other type of curve than a hyperbola. They could be cycloids or catenaries or some shape without an equation. Furthermore, the notion that a highly curved surface could be made up of straight lines is both marvelous and mysterious, and can lead to some amazing applications. Consider the example in Figure 7 ([3], p.216) where hyperboloidal gears allow drive shafts to meet at angles other than perpendicular or parallel.

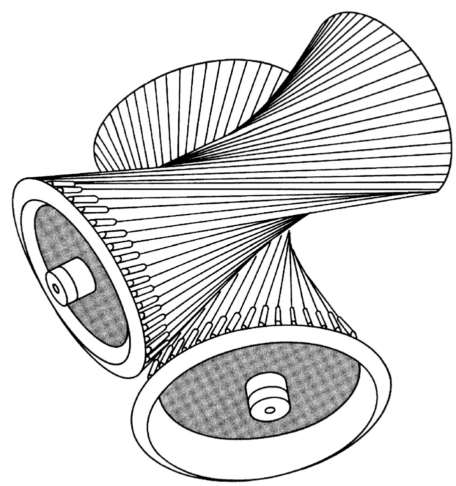

Figure 7 Hyperboloidal gears transmit motion to a skew shaft

References.

[1] Based on diagrams from the website https://sites.google.com/site/simeonlapinbleu/hyperboloid (Retrieved 12/12/2012)

[2] Kühnel, Wolfgang, *Differential Geometry: Curves – Surfaces – Manifolds*, 2nd Edition, Student Mathematical Library, Vol. 16, American Mathematical Society, 2006.

[3] Gardner, Martin, "Hyperbolas," in *Penrose Tiles to Trapdoor Ciphers ... and the return of Dr. Matrix*, rev, MAA, 1997, pp.205-217
