Hexagon-Rectangle Problem

(20 January 2019)

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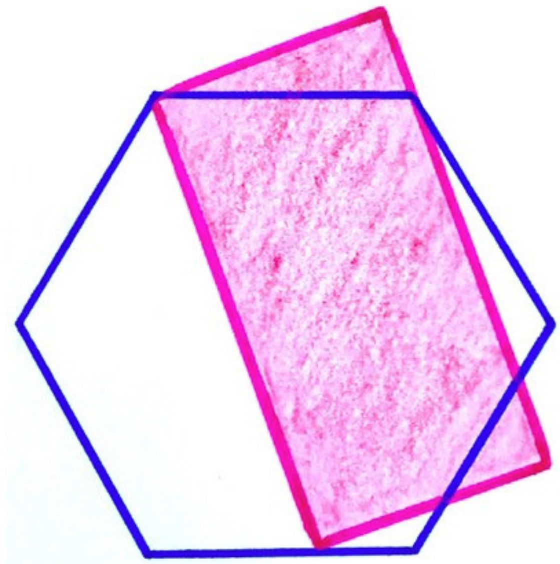
This is another interesting problem from Catriona Shearer. She shows the following figure with a regular hexagon and rectangle. "The area of the regular hexagon is 30. What's the area of the rectangle?"



(https://twitter.com/Cshearer41/status/1086589126718181376, retrieved 1/20/2019)

Solution

The first thing to notice is that Catriona did not indicate any constraints on the rectangle other than its upper left corner should coincide with a vertex of the hexagram, its right side should pass through the adjacent vertex of the hexagram, and the lower left corner of the rectangle should lie on the bottom side of the hexagram. Under such circumstances Polya suggests choosing a rectangle that satisfies these constraints which is the easiest to use, namely a vertical rectangle. Figure 1 shows

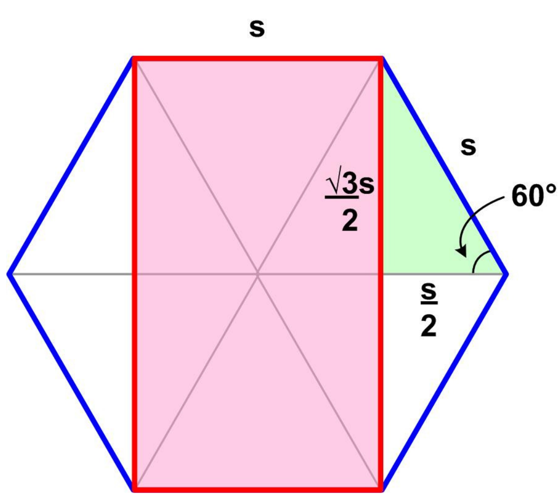


Figure 1 Vertical Rectangle with Annotations

such a configuration where we have labeled the side of the hexagon s. From the annotations we see that the area of the rectangle is

$$\text{Area of Rectangle} = \sqrt{3}s^2$$

The area of the entire hexagon is

$$\text{Area of Hexagon} = \text{Area of Rectangle} + 4 \times \text{Area of Triangle} = \sqrt{3}s^2 + 4(\sqrt{3}s^2/8)$$

$$\text{Area of Hexagon} = 3\sqrt{3}s^2/2$$

Therefore the ratio (Area of Rectangle) / (Area of Hexagon) = 2/3, which implies the area of the rectangle is (2/3) 30 = 20.

Area of Rotated Rectangle

Now the solution really is not complete unless we verify that any rotated version of the rectangle that satisfies the constraints has the same area. Notice that the dimensions of the rotated rectangle are not the same as the original.

From Figure 2 we see that the area of the rectangle is given by lw, where $l \cos \theta = \sqrt{3}s$ and $w = s \cos \theta$. Therefore,

$$\text{Area of Rectangle} = lw = (\sqrt{3}s/\cos \theta)(s \cos \theta) = \sqrt{3}s^2$$

which is exactly the value for the original rectangle. So the rotated rectangle does preserve the area.

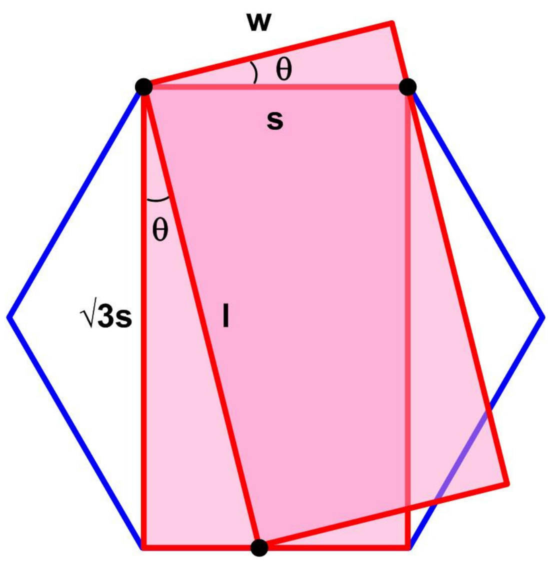


Figure 2 Rotated Constrained Rectangle