Complex Numbers – Geometric Viewpoint

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How we could have gotten here. We could arrive at the notion of complex numbers via the historical path of how to solve polynomial equations in real numbers. This would entail our having introduced the notion of extending our rational number system (field) by adding irrational numbers, such as $\sqrt{2}$, and showing we still have all the properties of a field. (The new numbers would be of the form a = p + q$\sqrt{2}$ where p and q are rational numbers.) But this piecemeal, one-at-a-time approach is inadequate (e.g., $\sqrt{3}$, another irrational number, does not belong to the field extension with $\sqrt{2}$, that is, there are no rational p, q such that $\sqrt{3}$ = p + q$\sqrt{2}$), and so we had to consider decimal expansions as a more encompassing way to obtain all the real numbers, thus leaving behind for the moment the idea of simple field extensions. In the course of solving polynomial equations in the reals, especially via the quadratic formula, we arrive at solutions involving the square root of negative numbers, in particular $\sqrt{-1}$. So we return to the idea first encountered with $\sqrt{2}$ by appending $\sqrt{-1}$ to the rest of the numbers (reals in this case) and showing we still have a number system with addition and multiplication and all the usual properties of a field.

Complex Number Definition

We shall define a **complex number** z to be of the form

$$z = a + i\,b$$

where a and b are real numbers and $i = \sqrt{-1}$, that is, $i^2 = -1$.[1] We call a the **real** part and b the **imaginary** part of z. We designated the set of real numbers by $\mathbb{R}$ (and the rationals by $\mathbb{Q}$, for quotients), so we shall designate the set of complex numbers by $\mathbb{C}$. Notice that when the imaginary part is 0, we only have a real number. So the reals $\mathbb{R}$ can be thought of as contained in the complexes $\mathbb{C}$ ($\mathbb{R} \subset \mathbb{C}$) in this way.

We now show that $\mathbb{C}$ satisfies all the properties of a number system (field) the same way we did when we added $\sqrt{2}$ to the rationals $\mathbb{Q}$. The critical property is that the addition, subtraction, multiplication, and division of two complex numbers is also a complex number. We treat i like any other number for addition and multiplication, using the fact that $i^2 = -1$.

If we set $z_1 = a + ib$ and $z_2 = c + id$, then

Addition: $$z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d) \in \mathbb{C} \quad (1)$$

Subtraction: $$z_1 - z_2 = (a + ib) - (c + id) = (a - c) + i(b - d) \in \mathbb{C} \quad (2)$$

Note, as with real subtraction, $z_1 - z_2 = z_1 + (-z_2)$

Multiplication: $$z_1 \times z_2 = (a + ib) \times (c + id) = (ac - bd) + i(ad + bc) \in \mathbb{C} \quad (3)$$

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[1] i certainly is not a real number, since the square of no real number can be negative. So this is a new beast, which we just tack onto the reals and see how far we can get using all the same operations as if everything were a real number. Recall that as far as the Greeks were concerned, $\sqrt{2}$ was a new beast in their day — which they ignored. That is, $\sqrt{2}$ was never a number (it was not rational), but rather the length of a line in a geometric figure: the hypotenuse of a right triangle with legs of length 1..

We will usually suppress the multiplication sign × and write $z_1 \times z_2 = z_1\, z_2$.

Division: (a) $$\frac{1}{z_2} = \frac{1}{c+id} = \frac{1}{c+id}\frac{c-id}{c-id} = \frac{c-id}{c^2+d^2} = \frac{c}{c^2+d^2} - i\frac{d}{c^2+d^2} \in \mathbb{C} \quad (4)$$

(b) $$\frac{z_1}{z_2} = \frac{a+ib}{c+id} = (a+ib)\frac{1}{c+id} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2} \in \mathbb{C} \quad (5)$$

As usual, we assume z_2 is not zero, which is equivalent to saying c and d cannot both be zero, and so $c^2 + d^2 \neq 0$.

With these expressions and the fact that $0 = 0 + i0$ and $1 = 1 + i0$ are still the additive and multiplicative identities in $\mathbb{C}$, it is easy to show $\mathbb{C}$ inherits all the field properties from $\mathbb{R}$.

Clearly this is far more complicated than adding and multiplying rationals. To help us understand the implications of these operations we turn to the geometric representation.

Geometric Representation

Just as we used a line to illustrate the behavior of integers, rationals, and then real numbers, so we turn to another geometric object, the plane. Every complex number $z = x + iy$ is determined by its real and imaginary parts x, y. These two real numbers can be used as the coordinates of a point in the plane (Figure 1). This is called the *rectangular coordinate* representation of the complex number z.

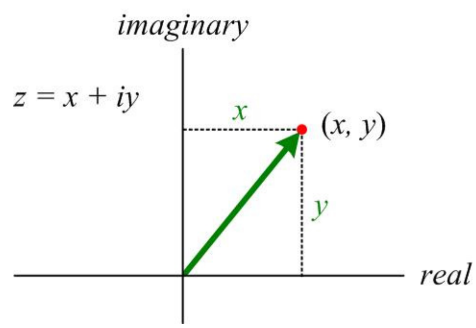

Figure 1 Rectangular Coordinates

As shown in the figure, we can also indicate the complex number by a vector with components x, y. We represented real numbers along the number line also by vectors, initially emanating from the zero point. The point in the plane where the real and imaginary axes cross, the origin, has coordinates (0, 0) and is the analog of zero on the real line.

Note also the fact that $\mathbb{R} \subset \mathbb{C}$ is represented geometrically by designating the horizontal axis as the real number line, which corresponds to complex numbers where the imaginary part y is 0.

Now let us look at what happens geometrically when we perform addition and multiplication of complex numbers.

Addition and Subtraction

Looking at Equation (1) for addition and interpreting the complex numbers as vectors we see from Figure 2 that adding two complex numbers involves the "head-to-tail" addition of vectors. That is, parallel translate the second vector until its tail coincides with the head of the first vector. Then the resulting sum is the new vector with tail coinciding with the tail of

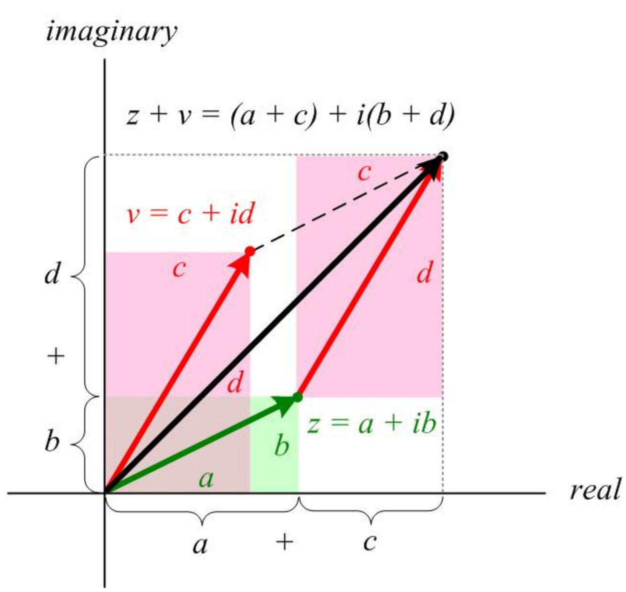

Figure 2 Complex Addition (Parallelogram Law)

the first vector and head coinciding with the head of the second. The picture of this operation forms the shape of a parallelogram and so is designated the *parallelogram law* of vector addition (mentioned by Newton in his *Principia*).

Since Equation (2) shows subtraction of complex numbers is the same as adding the negative of the second to the first, we can use the parallelogram law for subtraction, where first we flip the second vector 180º and add to the first as before.

Multiplication and Division

Multiplication by *i*. The expression for multiplication in Equation (3) is bad enough, but the expression for division in Equation (5) appears impenetrable. Let's first take a simple case of multiplying by *i.* Figure 3 shows that multiplication by *i* rotates the vector representation of the complex variable 90º counterclockwise. (We tip the green rectangle over on its long side to yield the red rectangle. Since the corners of a rectangle are all 90°, this means the corresponding diagonal was also rotated 90°.) So a second multiplication rotates 90º more or 180º in all. But that is consistent with $i^2 = -1$. That is, multiplying a vector by -1 is equivalent to flipping the vector 180° which corresponds to the negative of the vector.

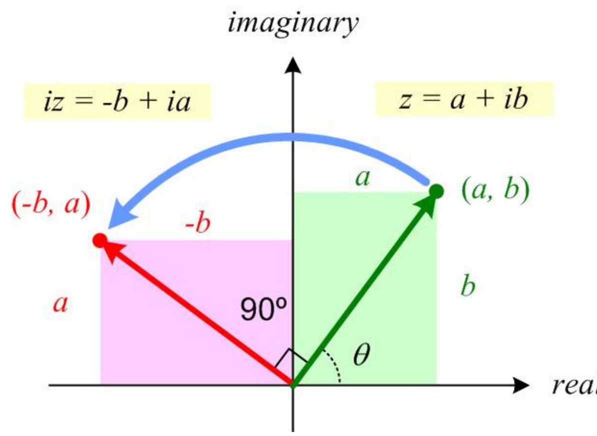

Figure 3 Multiplication by *i*

It is remarkable to notice the parallels. When we introduced -1 to generate the negative integers and extend the counting numbers to all the integers, we reduced the strangeness of multiplying a number by -1 to flipping the vector representing the number's position on the number line (a 180° rotation). Similarly, we are introducing *i* and adding it to the reals to obtain the complex numbers, and again multiplying a complex number by this new number *i* is equivalent to another rotation, this time 90°. Moreover, as before the multiplication by *i* includes the previous multiplication by -1.

General complex multiplication. But now we need to understand what happens when we multiply by any complex number and not just *i*. In order to get a picture of what might be happening, we need to consider yet another representation for a complex number in the plane. It is called the *polar coordinate* representation of the complex number *z* and is shown in Figure 4. In some ways it more closely captures the vector representation since it assigns a length and direction to the complex variable. From Figure 4 we see that the point (x, y) corresponding to the complex variable z is a distance r from the origin and the line from the point to the origin makes an angle θ with the real axis. These two numbers uniquely determine the point and are called its *polar coordinates*.

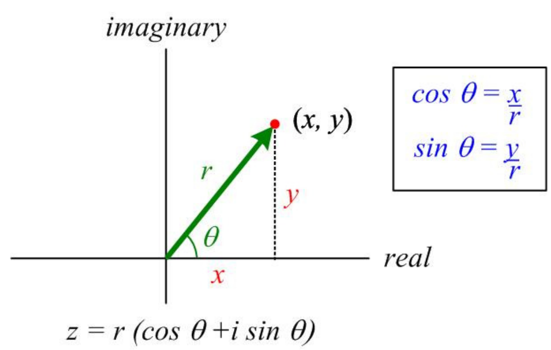

Figure 4 Polar Coordinates

For a complex variable z there is some additional terminology. $|z| = r = \sqrt{x^2 + y^2}$ is called the *modulus* of the complex variable z. $\arg z = \theta$ is called the *argument* of the complex variable z.

Unfortunately, in order to go back and forth between rectangular and polar coordinates, we need to introduce some concepts from trigonometry. We shall try to keep it to a minimum. As shown in Figure 4, the relationship between the two coordinate systems is via the two trigonometric functions sine and cosine of the angle θ and defined by the equations $cos\ \theta = x/r$ and $sin\ \theta = y/r$. The more direct way to show the transformation is

$$\left.\begin{array}{l} x = r\, cos\ \theta \\ y = r\, sin\ \theta \end{array}\right\} \quad \text{Rectangular to Polar Coordinate Transformation} \qquad (6)$$

From the Pythagorean Theorem we have

$$r^2 = x^2 + y^2 = r^2\, cos^2\ \theta + r^2\, sin^2\ \theta = = r^2\,(cos^2\ \theta + sin^2\ \theta)$$

which implies

$$cos^2\ \theta + sin^2\ \theta = 1$$

In order to "hide" the trig functions and to emphasize r and θ, we shall write the polar coordinate representation of a complex variable z as

$$\boxed{z = r\,\mathrm{E}(\theta) \ \text{ where } \ \mathrm{E}(\theta) = cos\theta + i\ sin\theta} \qquad (7)$$

Now we are ready to address complex multiplication. Let $z_1 = a + ib = r_1\,\mathrm{E}(\theta_1)$ and $z_2 = c + id = r_2\,\mathrm{E}(\theta_2)$, then

$$z_1\, z_2 = r_1\,\mathrm{E}(\theta_1)\, r_2\,\mathrm{E}(\theta_2) = r_1\, r_2\,\mathrm{E}(\theta_1)\mathrm{E}(\theta_2)$$

Now

$$\mathrm{E}(\theta_1)\mathrm{E}(\theta_2) = (cos\ \theta_1\ cos\ \theta_2 - sin\ \theta_1\ sin\ \theta_2) + i\,(sin\ \theta_1\ cos\ \theta_2 + cos\ \theta_1\ sin\ \theta_2)$$

This looks formidable, but in fact it represents a basic trigonometric identity, derived in Figure 5.

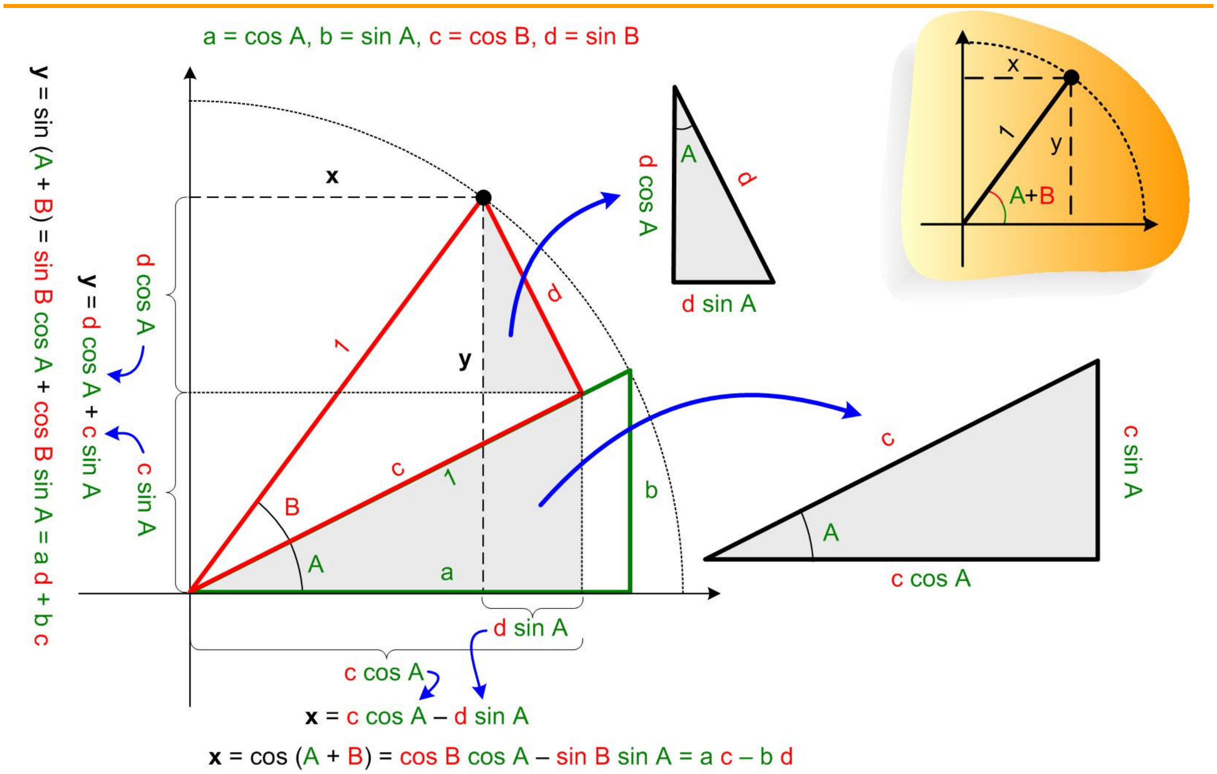

Figure 5 Proof of Trigonometric Sum of Angles Identities

Namely,

$$E(\theta_1)E(\theta_2) = \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) = E(\theta_1 + \theta_2)$$

A function that satisfies

$$\boxed{E(\theta_1)E(\theta_2) = E(\theta_1 + \theta_2)} \qquad \textbf{Exponential Property} \qquad (8)$$

is said to satisfy the *exponential property*. (This is made more explicit below on p.6.)

We can view the multiplication by i in terms of polar coordinates as follows.

$$i = 0 + i\,1 = \cos 90° + i \sin 90° = E(90°)$$

So for any complex number $z = r\,E(\theta)$,

$$i\,z = E(90°)\,r\,E(\theta) = r\,E(90° + \theta)$$

which is a counterclockwise rotation of 90° of the original complex variable z, as shown before.

In general we have for complex multiplication

$$\boxed{z_1 z_2 = r_1 r_2 E(\theta_1)E(\theta_2) = r_1 r_2 E(\theta_1 + \theta_2)} \qquad (9)$$

Complex multiplication, therefore, involves multiplying the moduli of the two numbers and adding their arguments, which amounts to rotating the vector associated with z_1 by an amount $\theta_2 = \arg z_2$ and changing the length (modulus) of z_1 by the multiple r_2 = modulus of z_2. In particular, $z^n = r^n E(n\theta)$. This is a bit difficult to visualize, so we will look at a number of examples.

Geometric examples. We shall consider some plots of complex polynomials, that is, polynomials of the form

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

where the coefficients $a_0, a_1, \ldots, a_n$ are complex numbers (possibly zero). If $z = r\,E(\theta)$, then $z^n = r^n E(n\theta)$ and $P(z)$ takes the form

$$P(z) = a_n r^n E(n\theta) + a_{n-1} r^{n-1} E((n-1)\theta) + \ldots + a_1 r\,E(\theta) + a_0$$

Consider the effect of adding each additional term in the polynomial. If $z = r\,E(\theta)$, then $z^n = r^n E(n\theta)$ means that not only does the power of a complex variable have a modulus of the same power, but as $z = r\,E(\theta)$ traverse a circle of radius r (θ varies from 0° to 360°), z^n whips around a circle of radius r^n n times. So we are essentially successively adding the effects of circles of increasing radii and rapidly turning arguments.

Figure 6 represents a plot of the quartic polynomial $P(z) = z^4 + z^3 + z^2 + z + 5$, where all the coefficients are real and equal to 1, except the constant term a_0 which is equal to 5. The values of z sweep out a circle of radius 1.5 and are represented in green in the plot. The corresponding P(z) values are shown in red. For large enough modulus $|z|$ the effect of the leading 4^{th} degree term is evident in the four loops in the plot as z makes one circuit. As $r = |z|$ shrinks, the loops coalesce into a curve looping more tightly around the constant term a_0 (5) represented by the large black dot in the plot. (Figure 7 shows the result of shrinking $|z|$ from 1.5 to 1.3)

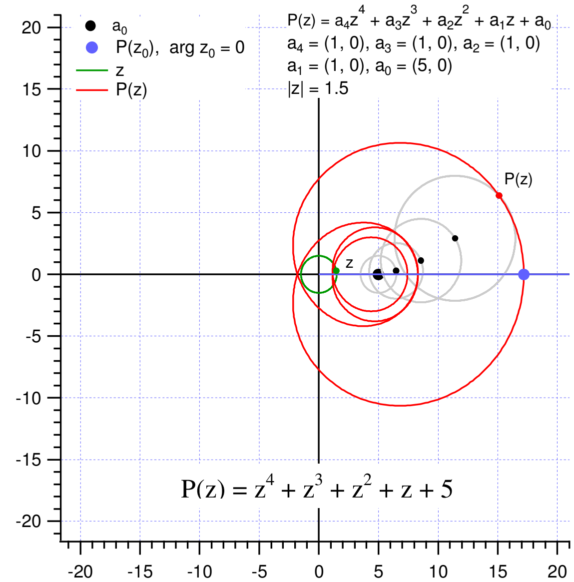

Figure 6 Plot of complex polynomial $P(z) = z^4 + z^3 + z^2 + z + 5$ for |z| = 1.5 and arg z ≈ 5°

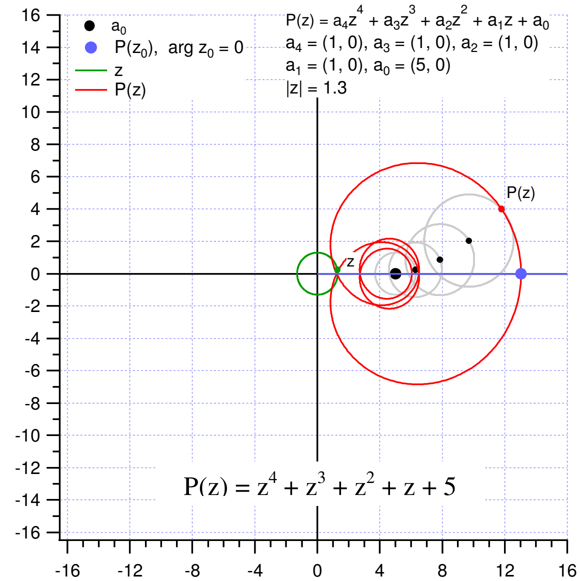

Figure 7 Plot of complex polynomial $P(z) = z^4 + z^3 + z^2 + z + 5$ for |z| = 1.3 and arg z ≈ 5°

A particular value of z was chosen with argument about 5°. It is the small green dot on the green z-circle. The trail of small black dots represent the values of each z^k term in the polynomial. They are shown residing on the (light gray) circle this term traverses k times as z traverses its circle. This z^k circle is centered on the value from the previous z^{k-1} circle for |z| = 1.5 and arg z ≈ 5°.

Figure 8 represents a similar plot of a complex cubic polynomial with one complex coefficient, namely, $P(z) = 2i\ z^3 + z^2 + 5\ z + 5$. A simpler point was chosen for the single evaluation, namely, |z| = 1.5 and arg z = 0°. This shows the simple progression of the successive terms in the polynomial as they are added. The effect of the multiplication by i is what we expect, namely, a counterclockwise 90° change in direction for the last term in polynomial.

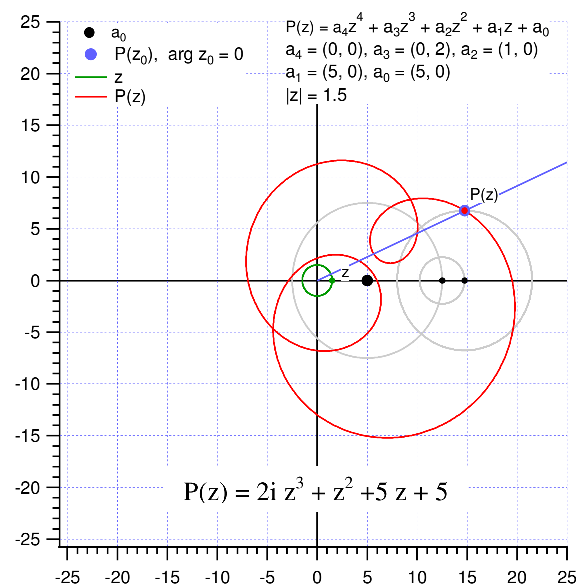

Figure 8 Plot of complex polynomial $P(z) = 2i\ z^3 + z^2 + 5\ z + 5$ for |z| = 1.5 and arg z ≈ 0°

Exponential Representation (Advanced)

Euler Formula. We are going to take the representation of a complex number given in equation (7), $z = r\ \mathrm{E}(\theta)$ where $\mathrm{E}(\theta) = cos\theta + i\ sin\theta$, a step further. From now on we will consider the angle θ given in radians instead of degrees, in order to be able to use the calculus. This means expressions such as $r\theta$ measure distance around a circle of radius r. Letting $\mathrm{dE}/\mathrm{d}\theta$ represent the derivative of E with respect to θ, we have

$$\mathrm{dE}/\mathrm{d}\theta = -sin\theta + i\ cos\theta = i\ \mathrm{E}(\theta)$$

Now we know from calculus with real variables that a function $y = \mathrm{f}(\theta)$ with derivative

$$dy/d\theta = a\,y$$

is of the general form

$$y = y_0\, e^{a\theta}$$

where y_0 is some constant, namely the value of y when $\theta = 0$. We shall assume $y_0 = 1$, since E(0) = 1. Then, setting a = i, it seems reasonable to *define*

$$\boxed{e^{i\theta} \overset{\text{def}}{=} \mathrm{E}(\theta)} \qquad (10)$$

The exponential property for E, equation (8), gives us the usual expression for exponentials

$$e^{i(\theta+\phi)} = \mathrm{E}(\theta+\phi) = \mathrm{E}(\theta)\,\mathrm{E}(\phi) = e^{i\theta}e^{i\phi}$$

So now the geometric representation for a complex number z, given in equation (7), becomes

$$\boxed{z = r\,e^{i\theta} \quad \text{where} \quad e^{i\theta} = \cos\theta + i\,\sin\theta \qquad \textbf{(Euler Formula)}} \qquad (11)$$

It turns out the legerdemain applied above to yield the Euler formula for the exponential is reinforced by complex power series. That is, if we take the usual real power series for e^x and substitute ix for x, we get the power series for $\cos x$ plus i times the power series for $\sin x$, namely the Euler formula. This approach involves totally different arguments and goes too far afield for the current paper, but the corroboration reinforces the reasonableness of the definition in equation (10).

Still this definition of $e^{i\theta}$ is a far cry from our original notion of exponentiation. It is amazing that it still preservers the properties we associate with exponentiation, but the physical meaning for the complex exponentiation is very different from the real version.

Complex Exponentiation. Since we have come this far we might as well take the next step. From calculus for real variables we have that the natural logarithm, $\ln x = \log_e x$, is the inverse function to the exponential function e^x, that is, $y = \ln x$ if and only if $x = e^y$. This means the Euler formula can be written $z = r\,e^{i\theta} = e^{\ln r}\,e^{i\theta}$. If we make the following definition,

$$\boxed{e^{x+iy} \overset{\text{def}}{=} e^x\,e^{iy}} \qquad (12)$$

then we have defined raising e to a complex power $z = x + i\,y$. Thus

$$w = e^z = e^x\,e^{iy} \text{ where } x = \ln|w| \text{ and } y = \arg w.$$

We can complete the circuit by defining the complex logarithm as the inverse function to e^z as

$$z = \log w = \ln|w| + i \arg w$$

Actually there are some issues here, since e^z is not one-to-one on the complex plane (it takes on the same values when multiples of 2π are added to its argument y). This leads to some fascinating developments where the complex plane is expanded to the notion of a Riemann surface. But that is more than enough for now.