Containing an Arc

(25 July 2018, rev 20 January 2019)

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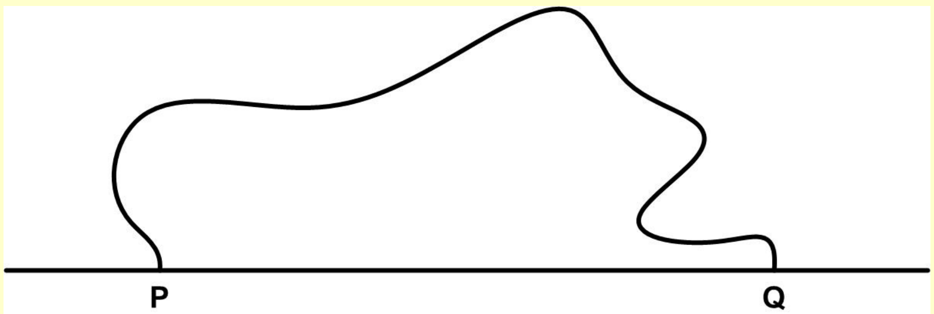
This problem from Futility Closet proved quite challenging.

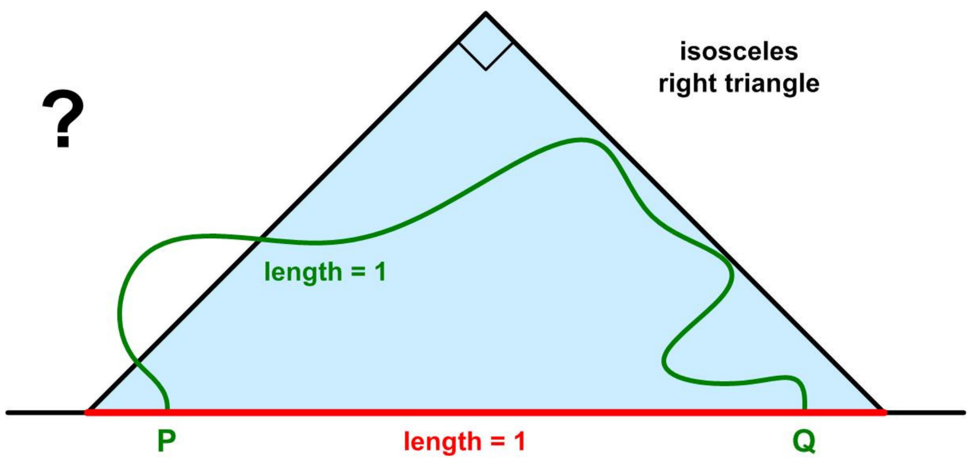
Containing an Arc

(https://www.futilitycloset.com/2018/07/24/containing-an-arc/, retrieved 7/25/2018)



University of Illinois mathematician John Wetzel called this one of his favorite problems in geometry. Call a plane arc *special* if it has length 1 and lies on one side of the line through its end points. Prove that any special arc can be contained in an isosceles right triangle of hypotenuse 1.

The following figure captures the problem statement:



The Futility Closet solution is quite ingenious, but it leaves some questions about its premises.

Futility Closet Solution

For special arc PQ, construct the smallest isosceles right triangle ABC with hypotenuse AB on line PQ that contains the arc. [See Figure 1] Let arc PQ touch the legs of this triangle at R and S. Now reflect arc PR in AC and arc SQ in BC, obtaining arcs RP' and SQ'. Now arc PRSQ equals arc P'RSQ' in length and lies between lines AP' and BQ', which are distance AB apart. The arc length is 1, and it's not less than AB, which proves the theorem. "Wetzel concludes, 'Pause and reflect!'"

(From Clayton W. Dodge, "Reflections of a Problems Editor," in Joby Milo Anthony and Howard Whitley Eves, *In Eves' Circles*, 1994.)

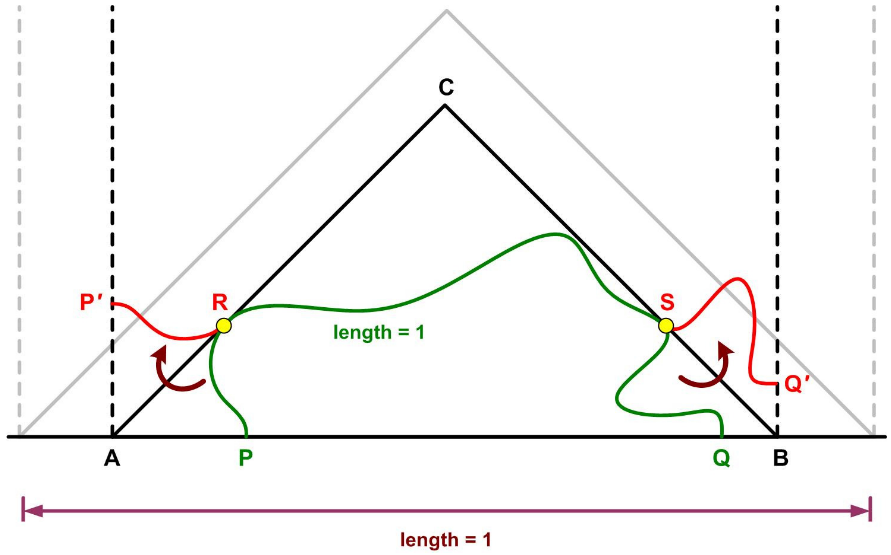


Figure 1 Futility Closet Solution

Comment

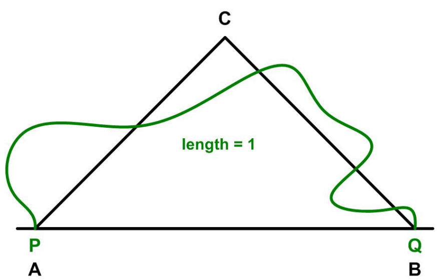


Figure 2 Initial Triangle

There is one issue that may be a bit vague. How do we know that we can find an isosceles right triangle that is tangent to both sides of the curve? Start with an initial isosceles right triangle with base the same as PQ (Figure 2). If the curve is inside this triangle, we are done, since the shortest distance between two points is a straight line, so the hypotenuse AB = PQ is shorter than the curve length of 1. Therefore, expanding the triangle so that AB = 1 would certainly contain the curve, since it already does.

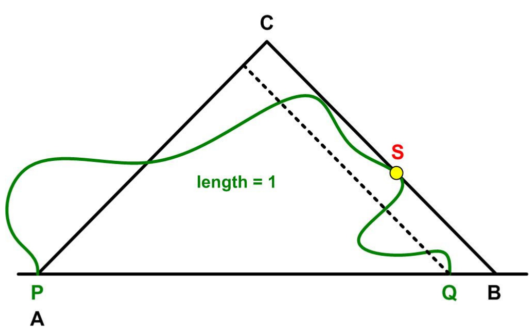


Figure 3 Expand to Contain Right Side

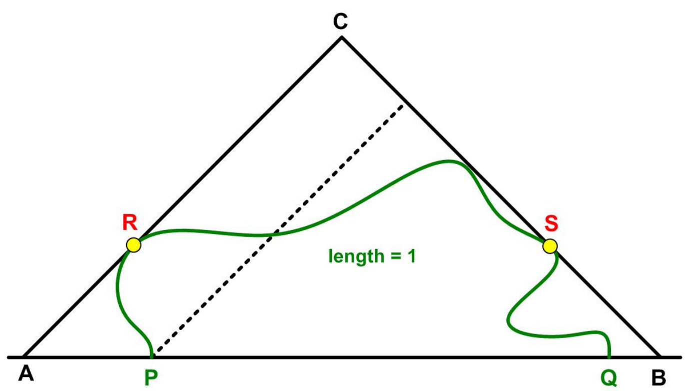


Figure 4 Expand to Contain Left Side

If the triangle does not contain the right side of the curve, expand the right leg by moving it in parallel until it just captures the curve on the right side at point S (Figure 3). Similarly, if the resulting triangle does not capture the curve on the left side, expand the left leg in parallel until it captures the left side of the curve at point R (Figure 4). So we are now in the situation for the Futility Closet solution with a triangle whose hypotenuse AB is less than the length of the curve and which contains the curve. So expanding this isosceles right triangle so that AB = 1 will still contain the curve and solve the problem.