

# Three Jugs Problem

4 August 2014, rev 17 October 2017

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Years ago (1967) I read about an interesting solution to the three jugs problem in a book by Nathan Court ([2]) which involved the idea of a billiard ball traversing a skew billiard table with distributions of the water between the jugs listed along the edges of the table. The ball bounced between solutions until it ended on the desired value. I thought it was very clever, but I really did not understand why it worked. Later I figured out an explanation, which I present here.

First, I give a statement of the problem (there are numerous versions, but this is the canonical one) and then describe the “billiard table solution.”

## Three Jug Problem

The classic “Three Jugs” problem: Two friends who have an eight-quart jug of water wish to share it evenly. They also have two empty jars, one holding five quarts, the other three. How can they each measure exactly 4 quarts of water? ([4])

The idea is that they can only fill or empty jugs. They cannot partially fill a jug, since there are no markings to aid them, other than the full amount level (Figure 1). This can be solved by trial and error, but there are some more organized approaches.

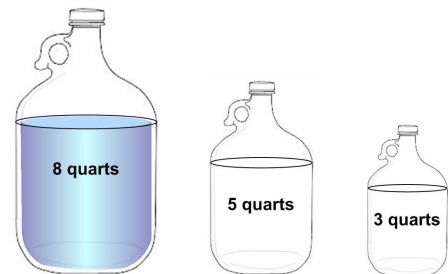


Figure 1 Three Jug Problem

The most ingenious solution is to map the problem onto a skew billiard table and plot trajectories of bouncing billiard balls under certain constraints.

## Two Jug Variant

We will begin with a two jug variant, as described in the movie *Die Hard III* as recounted by Ross and Polster in their column *Mathsnacks* for the teacher’s magazine *Viniculum* published by the Mathematical Association of Victoria, Australia ([10]) (slightly edited):

In the movie *Die Hard: With a Vengeance*, John (Bruce Willis) and Zeus (Samuel Jackson) are given a 5 gallon jug and a 3 gallon jug; then (with 30 seconds to think before they are blown to smithereens), they are ordered to use a fountain to fill the larger jug with exactly 4 gallons of water, using nothing but the jugs.

**Solution:** The points in the grid [Figure 2] correspond to the different ways in which the two jugs can be filled with water. The horizontal lines represent the filling or emptying of the 5 gallon jug; the lines of positive slope represent the filling or emptying of the 3 gallon jug; and the lines of negative slope represent the transferring of water between the jugs. Then, one solution to our problem is traced by the blue path: (5,0)-(2,3)-(2,0)-(0,2)-(5,2)-(4,3). So, you start by filling the 5 gallon jug from the fountain, then you fill the 3 gallon jug from the 5 gallon jug (leaving 2 gallons in the 5 gallon jug), then you empty the 3 gallon jug, etc.

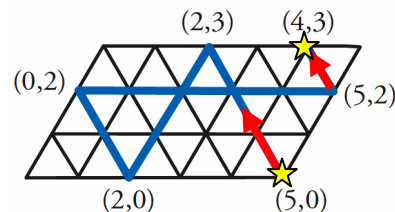


Figure 2 Billiard Ball Solution

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How this relates to the original three jug problem is that the fountain represents the 8 gallon jug (where quarts have been changed to gallons). Filling a jug from the fountain or emptying it into the fountain is equivalent to filling or emptying it into the 8 gallon jug. The problem involves no more than 8 gallons of water.

**Billiard Table Solution.** The solution involves using a “skew” billiard table in the form of a parallelogram and sending a billiard ball bouncing off the sides until it arrives at the desired point. All the angles in the parallelogram billiard table of Figure 2 are  $60^\circ$  so that the triangles are equilateral and therefore equiangular. Since a ball hitting the side of the table at an angle of  $60^\circ$  to the table edge will be reflected with the same angle, the trajectory of any ball will follow the edges of the triangles. There is a further assumption that the sides of the table are integer values, in this case 5 and 3, and that the ball starts at one of the integral points.

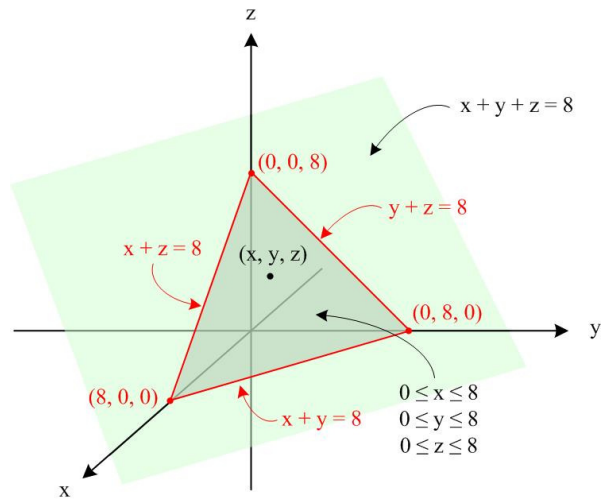
**Question: Why does the billiard table solution work and how can it address a three jug problem?**

### Solution

Let's return to the three jug problem involving 8, 5, and 3 quart jugs. Let  $x$ ,  $y$ , and  $z$  represent the amount of water in each of those jugs respectively at any given time. Then, since we are only using 8 quarts over all and not throwing any away, we always have the relation

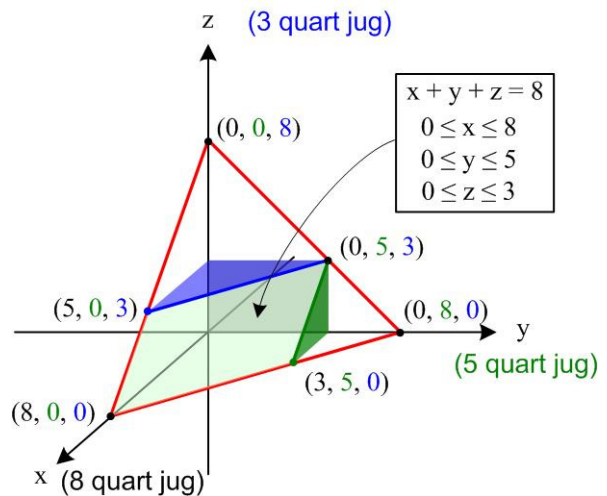
$$x + y + z = 8$$

From geometry we see this represents the equation of a plane in three dimensional space that cuts each axis 8 units from the origin (light green region in Figure 3). Furthermore, since we can't have negative water, all the values of  $x$ ,  $y$ , and  $z$  are  $\geq 0$ , which restricts the  $(x, y, z)$  coordinates to the equilateral triangle shaded in the figure.



**Figure 3** Plane for  $x + y + z = 8$

But only  $x$  (representing the maximum amount of water in the 8 quart jug) can reach the value of 8.  $y$  has a maximum value of 5 and  $z$  a maximum value of 3. This limits the coordinates to a truncated version of the shaded light green triangle in the form of a parallelogram (Figure 4). This is achieved by slicing the light green triangle with one (green) plane parallel to the  $xz$ -plane cutting the  $y$ -axis at 5, and another (blue) plane parallel to the  $xy$ -plane cutting the  $z$ -axis at 3. Since the green plane is perpendicular to the  $y$ -axis at  $y = 5$ , all  $(x, y, z)$  coordinates in that plane have  $y$  value 5. For those coordinates on the  $x + y + z = 8$  plane (light green triangle) this represents the situation where the 5 quart jug is full and the 8 and 3 quart jugs share the remaining 3 quarts between them.



**Figure 4** Truncated Triangle

There are several things to note in this figure

### Three Jugs Problem

that correspond to the physical actions of emptying and filling jugs. Each action only involves two jugs, leaving the water in the third jug unchanged. This corresponds to points moving in a plane for the two jugs which is perpendicular to the axis corresponding to the third, unchanged jug. Suppose the 5 quart jug (y-axis) is holding 4 quarts, and the remaining 4 quarts of water are divided between the 8 and 3 quart jugs (x and z values respectively) along a line that lies in a plane perpendicular to the y-axis cutting it at  $y = 4$  and satisfying  $x + z = 4$ . That means this plane slices the light green triangle along a line parallel to the right and left edges of the parallelogram (Figure 5). A similar discussion shows that pouring water between the 8 quart and 5 quart jugs (x and y values) represents points lying along lines parallel to the top and bottom edges of the parallelogram, since the amount of water in the 3 quart jug (z value) is left unchanged. (And pouring water between the 5 and 3 quart jugs (y and z values) leaving the 8 quart jug unchanged is represented by a line of points parallel to the right hand edge of the original light green triangle (red lines in Figure 6).)

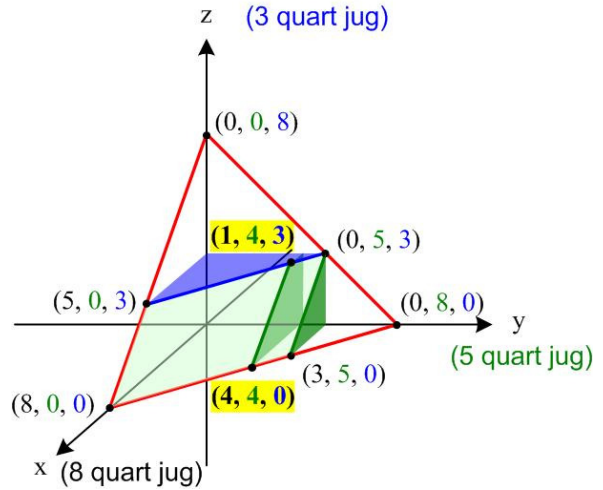


Figure 5 Planar slice

The second thing to notice is that because we are filling or emptying jugs and not stopping at intermediate levels, all actions involve arriving at points along the edges of the parallelogram in Figure 4 (also see Figure 5).

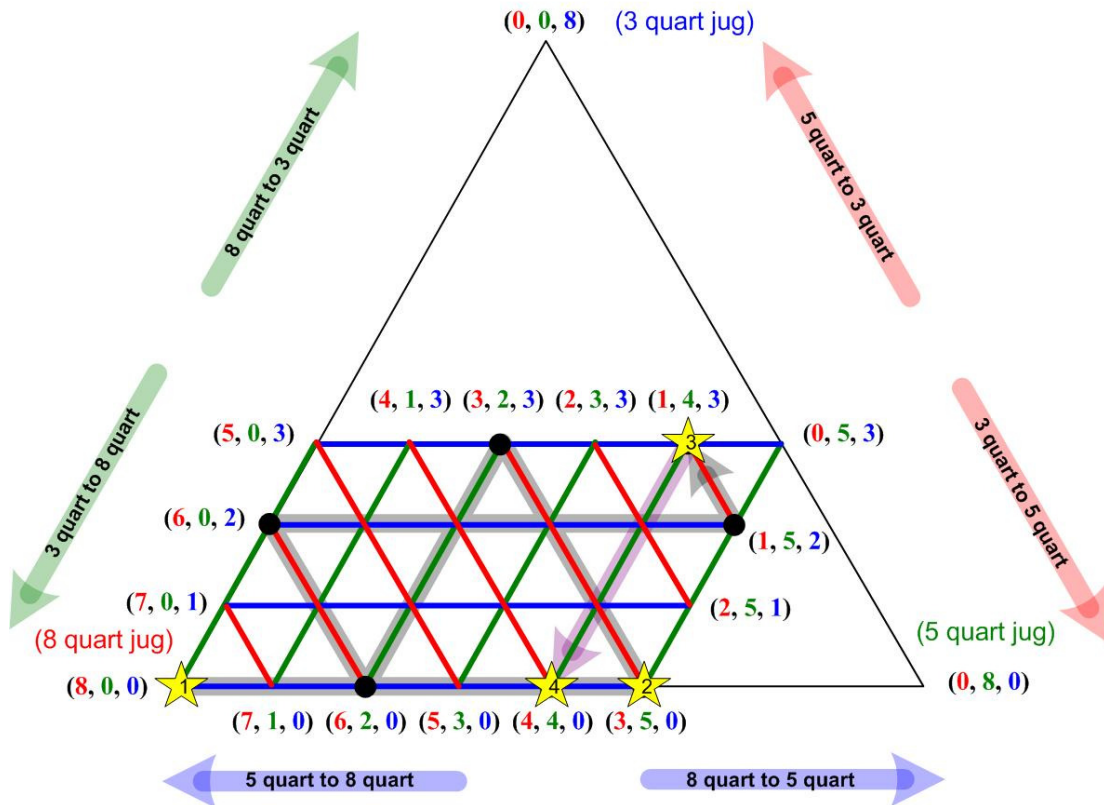


Figure 6 "Billiard Table" solution to Three Jugs Problem

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So in Figure 6 we finally arrive at the “Billiard Table” picture corresponding to Figure 2. Since we are using integral values for the size of the jugs and since we must fill or empty a jug, we only have integral values for the  $(x, y, z)$  coordinates along the edges representing the states of the jugs at any given time. Given that the original triangle from the plane  $x + y + z = 8$  was equilateral, and given that all the lines of coordinates representing the values of water in the jugs lie along lines parallel to the sides of the triangle, all angles are  $60^\circ$  and so can be interpreted as “reflections” of a billiard ball traversing the “table” if it starts at  $60^\circ$  angle initially. Actually, there is some ambiguity at corners of the table as far as the reflection interpretation goes.

Among other things, Figure 6 shows the solution of the two jug problem in Figure 2 starting at the star #2 and ending at the star #3. But rather than get the water from the fountain we should get it from the 8 gallon jug, and so start at star #1 and then go the star #2, which is filling the 5 gallon jug from the 8 gallon one.

Now the solution to the original, canonical Three Jug Problem requires one more step by going from start #3 to star #4 in Figure 6 (emptying the 3 gallon jug into the 8 gallon jug) and ending at point  $(4, 4, 0)$  with 4 gallons in the 8 gallon jug and 4 gallons in the 5 gallon jug.

### 7 Quart Max Jug

We illustrate two other versions of the Three Jug Problem where the largest jug is less than the sum of the other two. Figure 7 shows the case for the largest jug being 7 quarts instead of 8. Since 7 is not an even number and we can only obtain integral values of quarts in the containers, it is clear we cannot divide the water into two equal parts. And having less than eight quarts of water available we cannot get two containers with 4 quarts in each. We could get two with 3 quarts each, or even 2 quarts each, as shown in Figure 7.

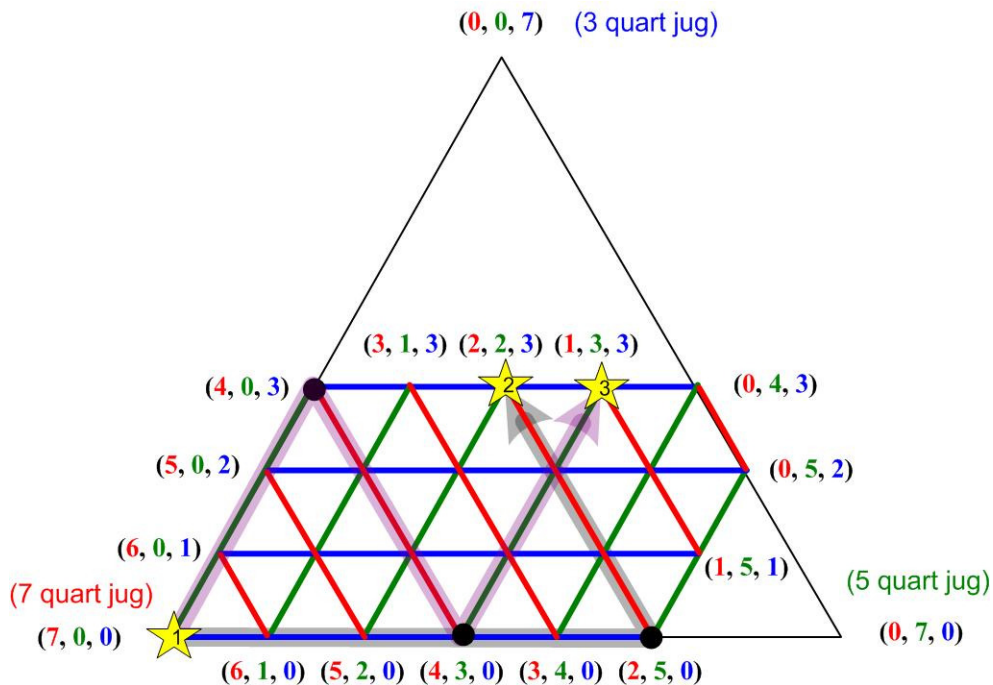


Figure 7 Three Jug Problem with 7qt, 5qt, 3qt jugs

### 9 Quart Max Jug

Figure 8 shows a case where the largest jug is greater than the sum of the other two, namely 9 quarts instead of 8 quarts. Again 9 is odd, so we cannot divide all the water evenly between two people, but we can consider getting 4 quarts in each of the two largest jugs. But that would be the

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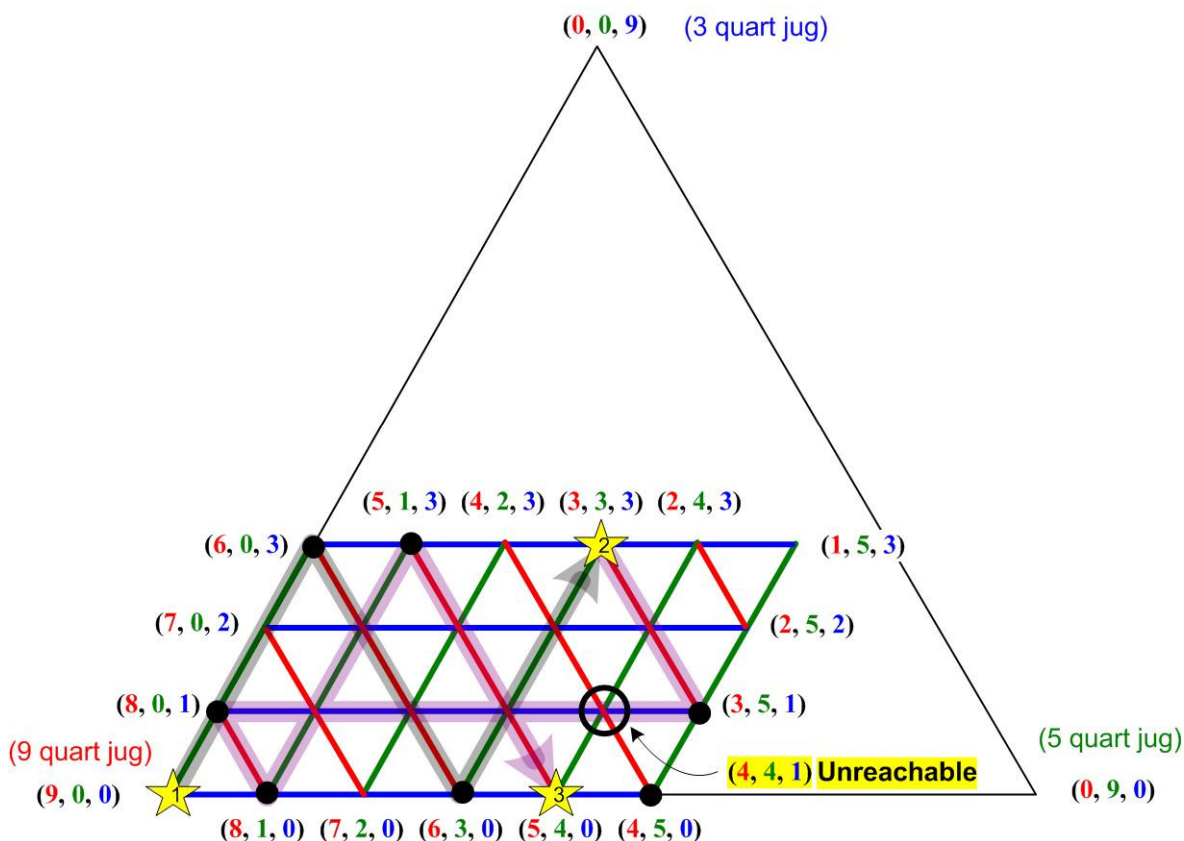


Figure 8 Three Jug Problem with 9qt, 5qt, 3qt jugs

coordinate  $(4, 4, 1)$ , which does not lie on the edge of the parallelogram. Therefore it is not reachable by the filling and emptying actions that we are allowed, and so we cannot achieve this solution.

Figure 8 does show we can divide the water equally between 3 people. The coordinate  $(3, 3, 3)$  lies on the edge of the parallelogram and the path reaching it is shown from star #1 to star #2. If we wanted to have 4 quarts in the 5 gallon can, we can achieve that by continuing the path from start #2 to star #3 at  $(5, 4, 0)$ .

### Other Pouring Problems

Dudeney ([6] p.152) has the following puzzle:

**410. DELIVERING THE MILK** A milkman one morning was driving to his dairy with two 10-gallon cans full of milk, when he was stopped by two countrywomen, who implored him to sell them a quart of milk each. Mrs. Green had a jug holding exactly 5 pints, and Mrs. Brown a jug holding exactly 4 pints, but the milkman had no measure whatever.

How did he manage to put an exact quart into each of the jugs? It was the second quart that gave him all the difficulty. But he contrived to do it in as few as nine transactions—and by a “transaction” we mean the pouring from a can into a jug, or from one jug to another, or from a jug back to the can. How did he do it?

This is clearly a pouring problem, but it involves 4 containers instead of 3, so it does not lend itself to the billiard table method (or the “trilinear coordinates” approach), and Dudeney did not use such a graphical method in his solution. However, Martin Gardner ([7] p.33) claims a “tetrahedral coordinates” approach to the four container problem was described by O’Beirne in his book ([9] Chapter 4).

## History

Dudeney (1917, [5] p.109) provides the following history for the “pouring problems” or “Tartaglia’s measuring puzzles.”

Apparently the first printed puzzle involving the measuring of a given quantity of liquid by pouring from one vessel to others of known capacity was that propounded by Niccola Fontana, better known as “Tartaglia” (the stammerer, 1500-1559). It consists in dividing 24 oz. of valuable balsam into three equal parts, the only measures available being vessels holding 5, 11, and 13 ounces respectively. There are many different solutions to this puzzle in six manipulations, or pourings from one vessel to another. Bachet de Méziriac reprinted this and other of Tartaglia’s puzzles in his *Problèmes plaisans et délectables* (1612). It is the general opinion that puzzles of this class can only be solved by trial, but I think formulae can be constructed for the solution generally of certain related cases. It is a practically unexplored field for investigation.

## Further Reading

Bellman and Cooke (1970, [1] p. 195) provide more recent background on some algorithmic and mathematical solutions:

The solution by means of the billiard ball method was first published (but not so named) in Tweedie (1939, [12]). The “billiard ball” interpretation is due to Y. I. Perelman, according to N. Court (1964, [2] p.199). Tweedie pointed out that the coordinates  $(x, y, z)$  describing the state of the jugs can be taken to be the “trilinear coordinates” of a point with respect to an equilateral triangle.

Gardner (1963, [7] pp.248-266, or 1967 pp.29-38), Court (1964, [2] pp.198ff.), Coxeter and Greitzer (1967, [3] pp.89ff), Michael (2009, [8] pp.148ff), and Stewart (2009, [11] pp.105, 299ff) all provide discussions of the billiard ball or trilinear coordinate solution.

## References

- [1] Bellman, Richard and Kenneth L. Cooke, *Algorithms, Graphs and Computers*, Academic Press, 1970.
- [2] Court, Nathan A., *Mathematics in Fun and in Earnest*, The Dial Press, 1958, Signet Science Library Books, 3d printing 1964, Dover Publications, 2006
- [3] Coxeter, Harold Scott Macdonald and Sammuel L. Greitzer, *Geometry Revisited*, Volume 19, Mathematical Association of America, 1967
- [4] Cut-the-Knot website, <http://www.cut-the-knot.org/ctk/Water.shtml>
- [5] Dudeney, Henry Ernest, *Amusements in Mathematics*, Thomas Nelson & Sons Ltd, 1917, reprint Dover Publications, 1958.
- [6] Dudeney, Henry Ernest, *536 Puzzles & Curious Problems (Modern Puzzles (1926), Puzzles and Curious Problems (1931))*, Martin Gardner, editor, Charles Scribner’s Sons, New York , 1967
- [7] Gardner, Martin, “Mathematical Games: How to solve puzzles by graphing the rebounds of a bouncing ball”, *Scientific American*, Vol. 209, No. 3 (September 1963), pp. 248-266. Also in *Martin Gardner’s 6<sup>th</sup> Book of Mathematical Diversions from Scientific American*, 1971, pp. 29-38
- [8] Michael, T.S., *How to Guard an Art Gallery and Other Discrete Mathematical Adventures*, The Johns Hopkins University Press, 2009
- [9] O’Beirne, T. H., *Puzzles and Paradoxes*, New York, Oxford University Press, 1965.

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- [10] Ross, Marty and Burkard Polster, "Mathsnacks: Perfect Puzzles," *Vinculum*, Vol 43, No 1, March 2006, Mathematical Association of Victoria, Australia, ([www.qedcat.com/mathsnacks/mathsnacks09\\_puzzles.pdf](http://www.qedcat.com/mathsnacks/mathsnacks09_puzzles.pdf), retrieved 7/29/13)
- [11] Stewart, Ian, *Professor Stewart's Hoard of Mathematical Treasures*, Profile Books Ltd., 2009, Basic Books, 2010
- [12] Tweedie, M. C. K., "A Graphical Method of Solving Tartaglian Measuring Puzzles," *Math. Gazette* 23 (1939), 278-282

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