South Dakota Travel Problem

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Reading a recent article about a school district in southeast South Dakota that had to reduce school attendance to four days a week took me to a Google map of the area (just southwest of Sioux Falls). As Figure 1 shows, the road network is startling. There are no diagonal roads of any significance (the long diagonal black line is a railroad and the shorter diagonal black line appears to be an abandoned railroad bed). The grid spacing between roads is 1 mile x 1 mile and is based on the Public Land Survey System (PLSS) which was used to establish parcels of land to be sold at public auction during the U.S. westward expansion and settlement. (See [1] for a detailed explanation.)

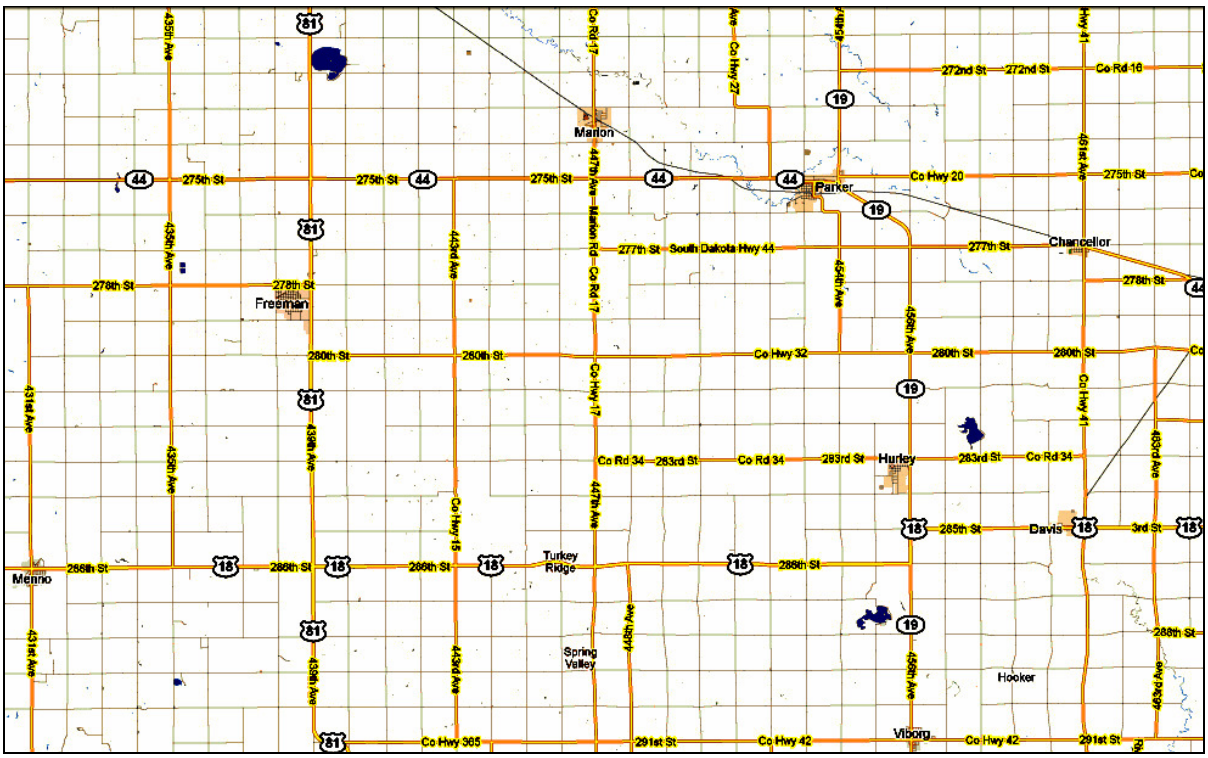

Figure 1 Google map of area just southwest of Sioux Falls, South Dakota

This road pattern immediately brought to mind the Taxicab Geometry, apparently first considered by Hermann Minkowski of Einstein Special Relativity fame. It gets its name from the fact that the distance between two points is measured as if they were intersections of perpendicular streets in a city where we can only move between the points by traveling along horizontal and vertical streets, like a taxicab. (References [2], [3], [4] are a few of the many available on the subject.) We shall only consider some simple ideas here.

Figure 2 provides some examples of measuring distance between two towns using the Taxicab distance compared with using the regular straight line or Crow-Fly distance. In the case of Hurley and Chancellor in the middle right of the map the Taxicab distance between the towns is 11 miles (5 mi east and 6 mi north). It does not matter if we try to go diagonally with a zig-zag path that is made up of horizontal and vertical legs; we will still travel 11 miles.

In the case of Freeman and Marion in the upper left of the map we see that the Taxicab distance is

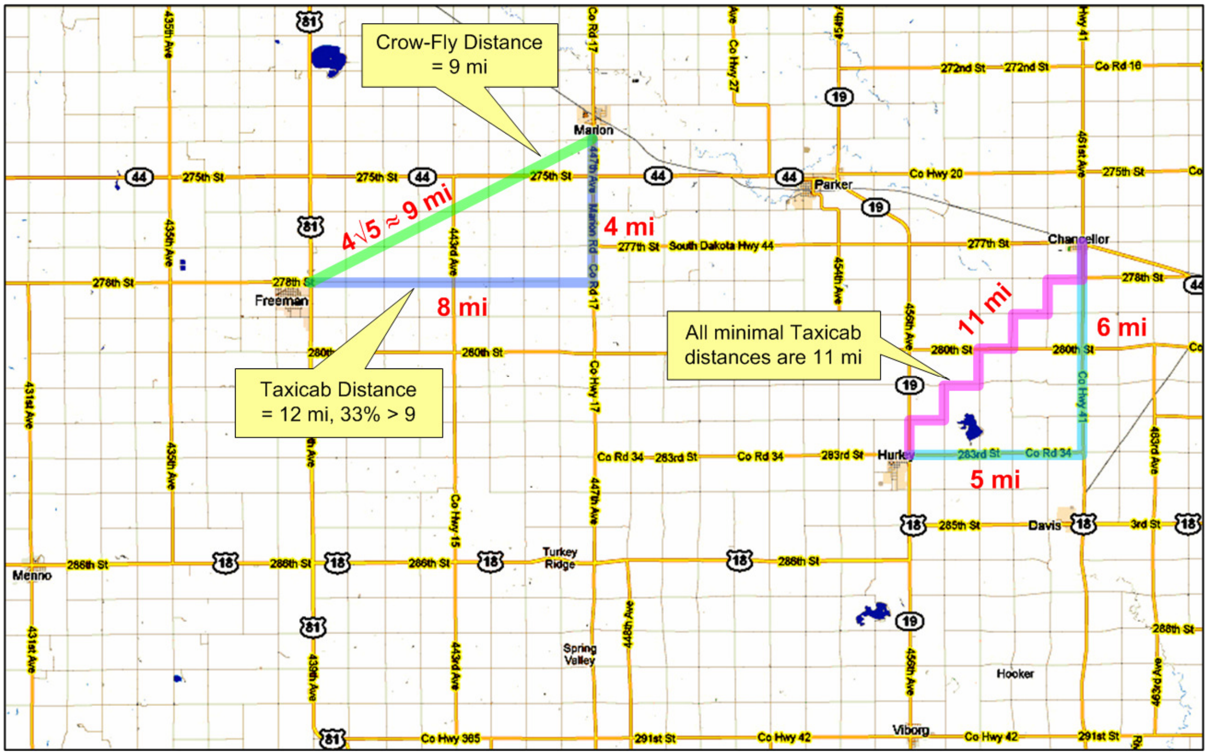

Figure 2 A comparison of Taxicab distances and "Crow-Fly" (Euclidean) distances.

12 miles (8 mi east and 4 mi north), whereas the straight line distance is about 9 miles (square root of the sum of the squares of the east and north values). The Taxicab distance of 12 miles is 33% longer than the straight line distance of 9 miles.

Question. So a natural question is what is the maximum percentage of the increase of the Taxicab distance between two points over the straight line distance? and in what angular direction would that maximum excess be?

Dakotas' "Coordinate System" Before we turn to the solution of that question it might be of interest to notice one further remarkable thing about the roads in South Dakota. There are some very large street numbers being used for the grid roads. For example 456th Ave runs north-south through Hurley and 439th Ave. runs north-south through Freeman. In the other direction 283d St runs east-west through Hurley and 278th St runs east-west through Freeman. In fact the east-west streets decrease by one number as we move north one grid space and the north-south avenues decrease by one number when we move west one grid space. Amazingly with these street and avenue numbers we can locate any town and measure the Taxicab distance between them. For example, the Taxicab distance between Hurley and Freeman is (456 – 439) + (283 – 278) = 17 + 5 = 22 miles.

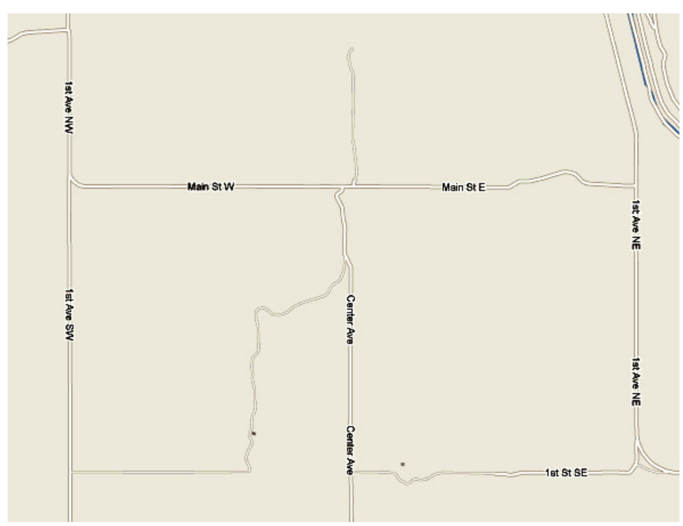

Figure 3 "Origin" of Dakotas' Street/Avenue Grid

I became curious to locate the (0, 0) origin of this "coordinate" system. It turns out it is located in the middle of *North* Dakota! The east-west street there is Main St. and the north-south avenue is Central Ave. (Figure 3) Perhaps the Dakotas can be used as a teaching aid for negative numbers and geometric coordinates!

Answer to Question:

The maximum percentage excess of the Taxicab distance over the straight line distance is 41.4% and that would occur when the straight line from the first point to the second makes a 45° angle with the horizontal.

Solution:

We can assume that we are measuring the distance from one point to another located north and east of the first. By symmetry the answer would be the same in all four quadrants. It just makes it easier to deal with positive numbers. Let x be the east-west distance from the first point and y be the north-south distance. The straight line distance is given by r, where $r^2 = x^2 + y^2$ and the Taxicab distance is given by x + y. The percentage (fraction) of excess with respect to the straight line distance is

$$\frac{(x+y)-r}{r} = \frac{x+y}{r} - 1 \qquad (*)$$

So for a given straight line distance r we want to find the x, y that maximize x + y. That will give the maximum ratio or percentage. Mathematically we are saying maximize z = x + y where x, y are constrained to satisfy $x^2 + y^2 = r^2$, that is, (x, y) are the coordinates of a point lying on a circle of radius r. In other words, where on the circle of radius r is z = x + y maximum?

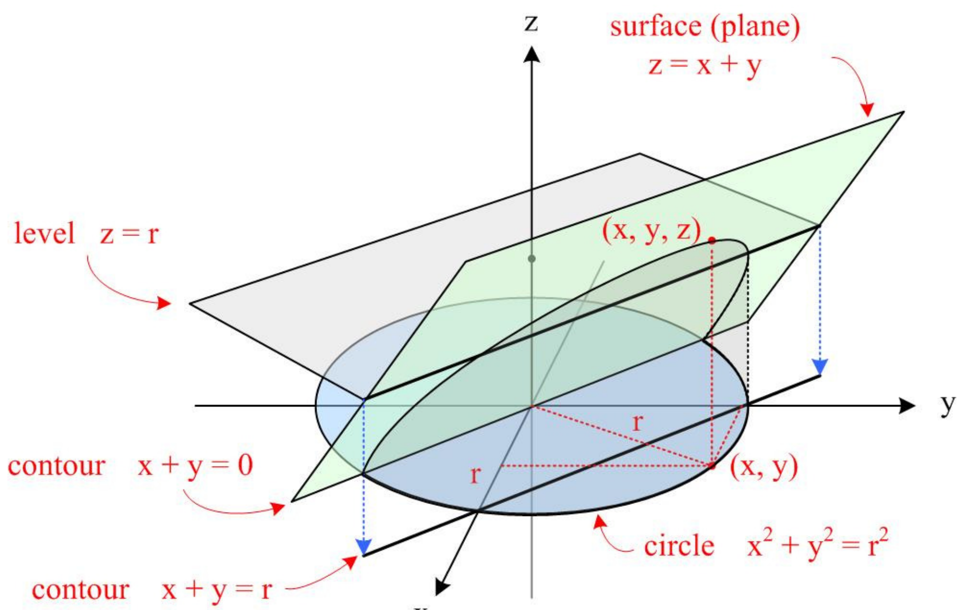

Figure 4 Finding the highest point (x, y, z) on the plane z = x + y over the circle of radius r

Figure 4 represents the geometry of the problem where the sum z = x + y rises over the xy-plane as (x, y) moves away from the origin in positive directions. We are only interested in the values of (x, y, z) where (x, y) is a point on the circle (and both x and y positive). It is easier to convert this to a two dimensional picture. We take constant elevation (z = constant) slices through the rising z = x + y plane. These intersections are horizontal straight lines that we can project down onto the xy-plane and label them with the elevation value. These lines are called constant height or elevation contours, which is what you see on a USGS topographic contour map.

Figure 5 shows the resulting two-dimensional contour plot superimposed over the circle of radius r. The 0 contour line is where the plane z = x + y intersects the xy-plane. The points on this contour clearly lie on the line y = - x through the origin (we are only interested in non-negative x, y values, however – those in the first quadrant). As we increase the sum (z = x + y), the constant height contours are successive parallel lines. As we walk around the circle, we can imagine climbing an inclined plane "mountain". Clearly, the highest point will be where the corresponding contour only touches the circle at one extreme point. This means the contour line is tangent to the circle and therefore perpendicular to the radius line. Notice that the parallel contour lines cut the x and y axes at equal distances from the origin, so that the triangle formed by the contour and the axes is a right isosceles triangle with corner angles of 45°. Because the radius is perpendicular to the tangent contour, the angle the radius makes with the x-axis is also 45°. That means the blue right triangle is also isosceles and so x = y at the max point.

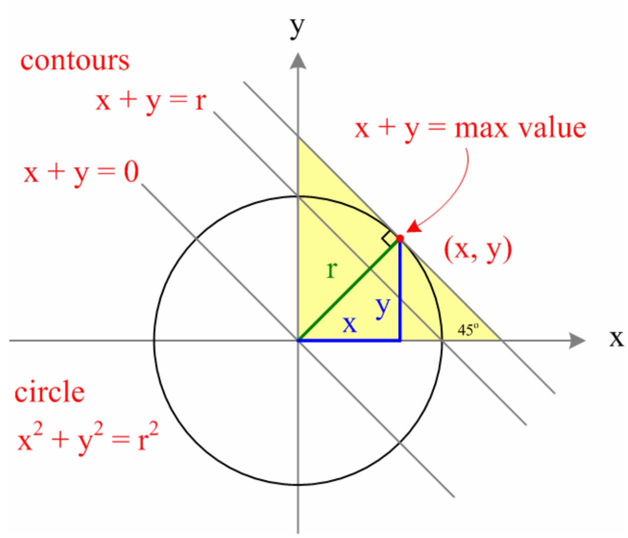

Figure 5 2-D Contour Plot

Since the (x, y) point lies on the circle, we get $r^2 = x^2 + y^2 = 2x^2$. Therefore $x = r / \sqrt{2}$. Substituting this value into equation (*) yields the maximum excess value of $\sqrt{2} - 1 \approx .414$ or 41.4%. And this maximum occurs when the line to the second point makes a 45° angle with respect to the horizontal (or equivalently, where the east and north distances are the same).

This certainly seems reasonable, since when the second point lies either due east or due north, there is no excess (the straight line distance equals the Taxicab distance). As we move away from due east or due north (that is, (x, y) on either the x-axis or y-axis) maintaining the constant straight line distance r, we see from Figure 5 that the sum increases above r to a max value ($= r\sqrt{2}$).

References

[1] "The Public Land Survey System (PLSS)," *nationalatlas.gov*, (http://nationalatlas.gov/articles/boundaries/a_plss.html, updated 1/26/11)

[2] "Taxicab geometry", *Wikipedia*, (http://en.wikipedia.org/wiki/Taxicab_geometry, updated 8/8/11)

[3] "Taxicab Geometry website!", (http://taxicabgeometry.net/, updated 2/4/11)

[4] Janssen, Christina, *Taxicab Geometry: Not the Shortest Ride Across Town*, MS MSS, Iowa State University, July 2007 (www.math.iastate.edu/thesisarchive/MSM/JanssenMSMSS07.pdf)