The Pythagorean Theorem

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All too frequently I come across the usual statements questioning why non-technical folks should bother studying math.[1] A typical example is the Pythagorean Theorem. People say, “What good is that? I’ll never use it. So why bother?” Ah, the famous “utility” argument – as if everything worthwhile must be “useful.” What about “useless” music? Why slave away learning how to play an instrument if you will never become a professional? And what is the *use* of listening to music? Even worse, what is the point of learning how to play checkers or chess, or card games? If you are not going to be a professional player or use them for gambling, what is the point? The point is, of course, that we *like* it. Music and game-playing enhance our lives in many ways that are not utilitarian. The mental challenge of games is especially satisfying, as is solving puzzles.

At its best, mathematics satisfies all these aspects – it is full of puzzles, mentally challenging, esthetically pleasing, and even quite utilitarian. I thought I would take this “useless” math example *par excellence* and show that, in fact, it harbors many of the best aspects of mathematics that anyone should find appealing.

First, what is the theorem? Figure 1 provides a picture. The statement is that for any right triangle (one with a 90° angle at one of its vertices), the sum of the squares of the lengths of the sides adjacent to the right angle, the legs, is equal to the square of the length of the side opposite the right angle, the hypotenuse.

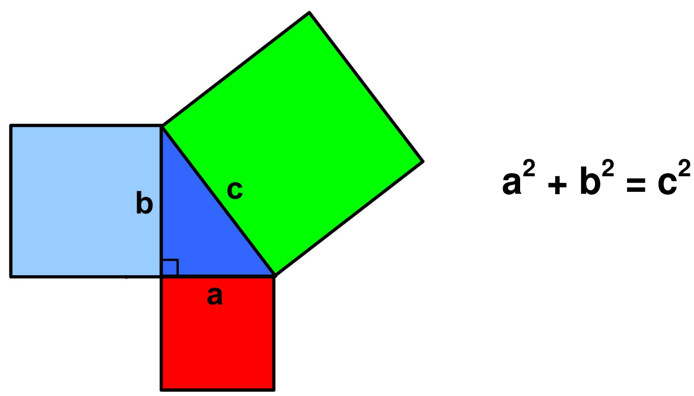

Figure 1 Pythagorean Theorem

There is some interesting history about this theorem that I won’t go into now.[2] But one thing I do want to emphasize is that even though it may be 2500 – or even 4500 – years old, it is *unchanged*. That is, it is as valid today as it was millennia ago. Very few things in life can say that. I always loved Shelley’s poem *Ozymandias* about the crumbled, stone ruins of the king who thought himself eternal, which ends with

“My name is Ozymandias, king of kings:
Look on my works, ye Mighty, and despair!”
Nothing beside remains. Round the decay
Of that colossal wreck, boundless and bare
The lone and level sands stretch far away.

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[1] A recent attack can be found in Andrew Hacker’s latest, *The Math Myth and Other STEM Delusions* (2015)

[2] See the website *Cut the Knot* by Alexander Bogomolny: http://www.cut-the-knot.org/pythagoras/, Remark 1

Mathematics is wonderfully eternal, and the amazing thing is that it survives, not in indelible media like stone and rock, but in the ephemeral domain of the human mind. It is marvelous how it endures down the years from the fragile human activity of passing it from one mind to another through education.

Pythagorean Theorem Proofs

So the puzzle aspect of the Pythagorean Theorem is how do we show it is true? Showing its validity through deductive reasoning is called a proof. Since the theorem is at least 2500 years old, and possibly older, it is not surprising that a plethora of proofs have been discovered through the years.[3] We shall consider a geometric version and then an algebraic version.

Geometric Proof

The idea is shown in Figure 2. We first construct a square $\mathbf{S_4}$ with sides a + b. We then place four copies of the right triangle **T** inside the square as shown in Figure 2(a). The space not covered by the 4 triangles is the same as the areas on the legs of the triangle, namely the medium light blue square $\mathbf{S_1}$ and the small red square $\mathbf{S_2}$. We then rearrange (translate and rotate) the four triangles as shown in Figure 2(b). Now the space not covered by the triangles is the large green square $\mathbf{S_3}$. Since the uncovered space must be the same in both cases, we have $\mathbf{S_1} + \mathbf{S_2} = \mathbf{S_3}$ or $b^2 + a^2 = c^2$. I saw this version of the proof years ago on Jacob Bronowski's television show *The Ascent of Man*. (See the Appendix p.7 for another, more direct geometric proof.)

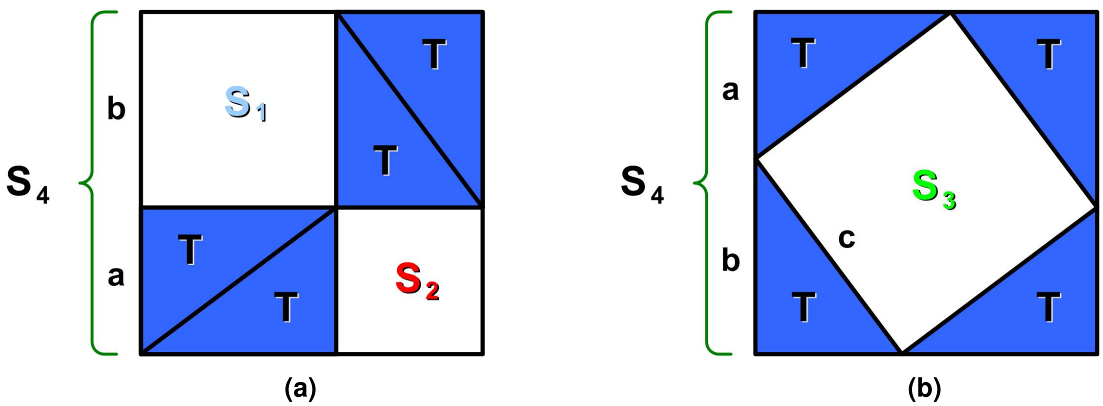

Figure 2 Pythagorean Theorem Geometric Proof

Algebraic Proof

We consider a purely algebraic proof based on Figure 2(b). Figure 3 shows the original right triangle (blue) with the large square on side **c** (green) inscribed in the larger square of side **a + b**. The remaining space in the larger square consists of three more triangles congruent to the blue triangle, and so each with the same area. The area of the larger square is

$$\text{Area Large Square} = (a + b)^2 = a^2 + 2ab + b^2 \qquad (1)$$

But adding up the areas of the inscribed green square and four triangles, we have

$$\text{Area Large Square} = c^2 + 4(ab/2) = c^2 + 2ab \qquad (2)$$

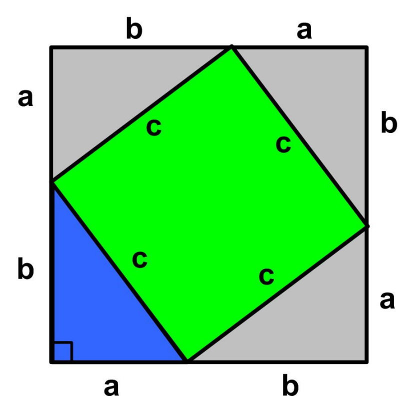

Figure 3 Pythagorean Theorem Algebraic Proof

[3] 116 of these proofs can be found at Bogomolny's website mentioned in Note 2

Therefore, equating the two expressions gives

$$a^2 + 2ab + b^2 = c^2 + 2ab \Rightarrow a^2 + b^2 = c^2$$

The steps in the algebraic proof actually mirror the geometric proof given above. That is, equation (1) corresponds to Figure 2(a) and equation (2) corresponds to Figure 2(b).

Plane Geometry

Both approaches above actually used properties of plane geometry, which we need to understand more clearly. Plane geometry is the study of the properties of figures drawn on a flat surface, a plane, which are unchanged when one figure is made to coincide with another by rigid motions, that is, by one or more translations (sliding), rotations, or flips (turning over). Two figures that can be made to coincide via rigid motions are said to be **congruent**. So plane geometry is the study of properties that remain the same for congruent figures.

Theorems are developed to say when two figures are congruent (can be made to coincide via rigid motions). For example, two triangles having their corresponding three sides equal in length are congruent. This is called the side-side-side (SSS) condition. So if two right triangles have corresponding legs of the same length, then their third sides, the hypotenuses, are also equal – by the Pythagorean Theorem! – and so the triangles are congruent.

Recall that the first geometric approach above relied heavily on the rigid motion of translation to slide the pieces of the smaller squares until they coincided with the larger square on the hypotenuse. And in the second approach we were using the SSS property to say the three grey triangles were congruent with original blue triangle. We then used the property that areas of congruent figures are the same to obtain our desired result.

So what happens when we are not on a flat, planer surface?

Geometry on a Sphere

We now wish to consider right triangles on a sphere and see what happens to the Pythagorean Theorem there. But first we need to figure out what "straight" lines on a sphere would mean, since we need to use those to construct triangles. We also need to make clear what an angle means as well, since we want the triangle to have a right angle in it.

Straight Line.

Assume you are standing on a sphere (like the earth) at some point **P** (see black arrow in Figure 4). Imagine a line running from your head to your feet and then passing through to the center of the sphere **C**. At your feet lay down a straight rod and then imagine a "straight" line passing through it without changing direction. The red arrow in Figure 4 represents the rod. You (black arrow) and the rod (red arrow) define a plane passing through the center of the sphere, which cuts the sphere in a circle. Any such circle whose center coincides with the center of the sphere is called a **great circle**. It seems reasonable to say that as long as you are walking along the red line, you are walking "straight ahead", that is, in a straight line. So we have our definition, namely, a **straight line** on the sphere is an arc of a great circle.

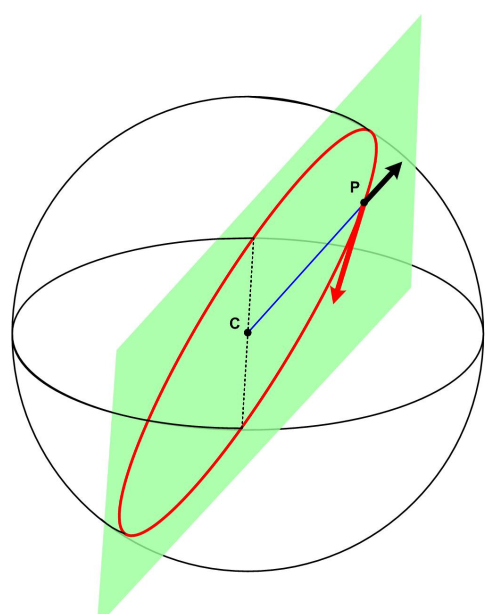

Figure 4 "Straight" Line on a Sphere

If we have two points on the sphere, we pass a plane through them and the center of the sphere. Then the straight line joining them will be the arc of the great circle between them. There is some ambiguity here, since there are usually two arcs to choose from, a shorter and then a longer going the other way around. We will always choose the shorter arc. If the two points are antipodal points (lying on opposite ends of a diameter of the sphere) then both arcs are the same length.

Shortest Distance

There is one other property of straight lines we are used to, namely, they are the shortest distance between two points on a plane. (This takes some discussion to make clear.) It turns out that arcs of great circles have that same property. (This is even harder to show. To see the effect, however, take a rubber band and stretch it between two points on a globe. The position it takes is along a great circle arc, and it is clearly the shortest distance.) Figure 5 provides an example of two points (New York City and Lisbon, Portugal) on the same latitude circle of a spherical earth whose great circle distance is 5405 km, but whose distance along the latitude circle is longer, namely, 5550 km. Latitude circles are obtained by slicing the earth with planes parallel to the equator. Except for the equator, whose plane passes through the center, all these latitude circles are *not* great circles. (There is some waffling in this discussion due to the fact that the earth is not a perfect sphere. In fact, it is closer to an oblate spheroid where the north and south poles are squashed a bit towards the center. But we shall continue to assume it is a sphere.)

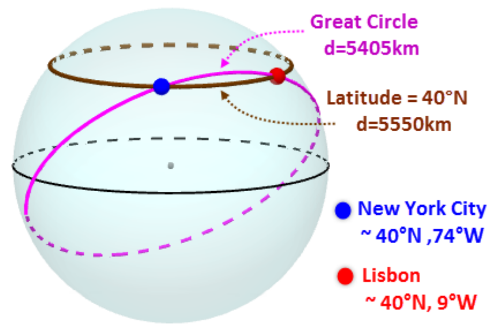

Figure 5 Great Circle vs. Latitude Circle

https://commons.wikimedia.org/wiki/File:Wiki_great_circle.png

Angles

One other thing we have to address is the meaning of an angle between two straight lines (great circle arcs) on a sphere. Suppose we have two lines (arcs) emanating from a point **P** on the sphere (see Figure 6). Extend the great circle arcs to the complete circles and consider the tangent lines to these circles at **P**, indicated by the two black arrows in Figure 6. These two arrows and their associated straight lines define a plane tangent to the sphere at **P**. We reduce the definition of angle between arcs to the plane geometry definition of an angle between two lines in a plane. So **angle θ between two great circle arcs** intersecting at a point **P** is the angle between the two tangent lines to the corresponding great circles intersecting at **P**.

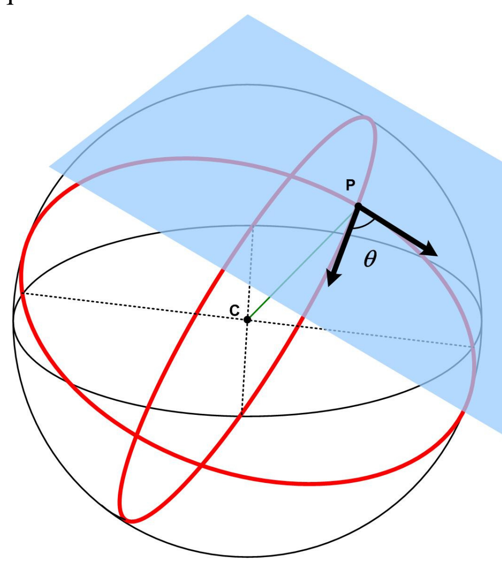

Figure 6 Angle Definition

Pythagorean Theorem on a Sphere?

We now want to draw a right triangle on a sphere and see if the Pythagorean Theorem holds. We are going to use the earth as our sphere and three cities to form the vertices of our right triangle. We will then measure the distances between them and see if they satisfy the Pythagorean Theorem.

In order to find a right triangle on earth we need to find a place where two great circles meet at right angles. Since all the meridians (constant longitude lines) are arcs of great circles, then any two points with the same longitude lie on a great circle arc defined by that meridian. As we saw in Figure 5, even though parallels (latitude circles) are perpendicular to meridians, only the equator is a great

Figure 7 "Right" Triangle in South America

circle. So we need to construct our earth triangle with one leg along the equator and another along a meridian. Figure 7 shows such a right triangle in South America using for vertices the cities Quito, Ecuador, Manapá, Brazil, and Porto Alegre, Brazil.

Table 1 shows the values for the locations of the cities obtained from Google Maps using the "What's here?" right click on each city. The values were given in signed decimal degrees. I added the values converted to deg-min-sec format. The last column records values for the distances

Table 1 Positions of Vertices of Right Triangle on Earth (Google Maps)

<table>
<tr><th>City</th><th>Latitude</th><th>Lat Dev</th><th>Longitude</th><th>Lon Dev</th><th>Distance*</th></tr>
<tr><td>Quito</td><td>-0.0651°
00° 04' S</td><td rowspan="2">0.1319°
≈ 8 mi</td><td>-78.5247°
78° 31' W</td><td></td><td>to Macapá
1895 mi</td></tr>
<tr><td>Macapá</td><td>0.0668°
00° 04' N</td><td>-51.0588°
51° 04' W</td><td rowspan="2">0.1758°
≈ 11 mi</td><td>to P. Alegre
2080 mi</td></tr>
<tr><td>Porto Alegre</td><td>-29.9732°
29° 58' S</td><td></td><td>-51.2346°
51° 14' W</td><td>to Quito
2735 mi</td></tr>
</table>

* http://www.mapdevelopers.com/distance_from_to.php

between the cities in miles based on computations from the Google Map Developers website.[4] Supposedly these numbers should all be consistent with one another, since they came from Google.

As we try to illustrate the abstract mathematics with a real example, we run into the inevitable mismatch. That is, real data never quite fits the general model we are using, so we have to measure the deviations from what we want and try to correct for them. Table 1 has two columns that show the deviations of the Quito-Macapá locations from lying on the equator (lat dev) and the Macapá-Porto Alegre locations from lying on a common meridian (lon dev). We make the assumption that a minute of lat dev and a minute lon dev at the equator is each approximately one mile (it is actually one nautical mile which is about 10% longer the a regular statute mile). In order to compensate for these deviations and to try to make the angle at Macapá closer to 90°, we will move the vertex at Macapá 11 miles west and 8 miles south. Thus the Quito-Macapá distance becomes 1884 miles and the Macapá-Porto Alegre distance becomes 2072 miles.

Now we compute the sum of squares of the legs to see how close it is to the square of the hypotenuse.

$$\mathbf{1884^2 + 2072^2 \approx 2800^2}$$

Then the difference is

$$\mathbf{2800 - 2735 = 65 \text{ miles.}}$$

The error is about

$$\mathbf{65/2735 \approx 2\%}$$

which is not very large.

Using location values from Wikipedia yields even greater deviations, as shown in Table 2. Still using the distances from Table 1, which may be erroneous with these new locations, we have the following computations.

$$\mathbf{1885^2 + 2064^2 \approx 2795^2}$$

$$\mathbf{2795 - 2735 = 60 \text{ miles.}}$$

$$\mathbf{60/2735 \approx 2\%}$$

Table 2 Positions of Vertices of Right Triangle on Earth (Wikipedia)

<table>
<tr><th>City</th><th>Latitude</th><th>Lat Dev</th><th>Longitude</th><th>Lon Dev</th><th>Distance*</th></tr>
<tr><td>Quito</td><td>00° 14′ S</td><td rowspan="2">≈ 16 mi</td><td>78° 31′ W</td><td></td><td>to Macapá
1895 mi</td></tr>
<tr><td>Macapá</td><td>00° 02′ 02″ N</td><td>51° 03′ 59″ W</td><td rowspan="2">≈ 10 mi</td><td>to P. Alegre
2080 mi</td></tr>
<tr><td>Porto Alegre</td><td>30° 01′ 59″ S</td><td></td><td>51° 13′ 48″ W</td><td>to Quito
2735 mi</td></tr>
</table>

* http://www.mapdevelopers.com/distance_from_to.php

Deviation from Pythagorean Theorem

So is this discrepancy between what the plane geometry form of the Pythagorean Theorem would say and what we measure real, or are there more errors that would cancel the difference? It is true

[4] http://www.mapdevelopers.com/distance_from_to.php

there are more unknowns in these calculations that might make some difference, but it seems that the discrepancy is real.

In fact, consider an extreme case, namely that shown in Figure 8. Here we have our right triangle with two vertices on the equator, one on the Greenwich Meridian (0°) and the other 90° W. The third vertex is at the North Pole. The Greenwich Meridian intersects the equator at a 90° angle, so the triangle is technically a right triangle. But the length **L** of each side of the triangle is the same, in fact, ¼ the circumference **C** of the globe. If we assume **C** is about 25,000 miles, then each **L** is about 6,250 miles.

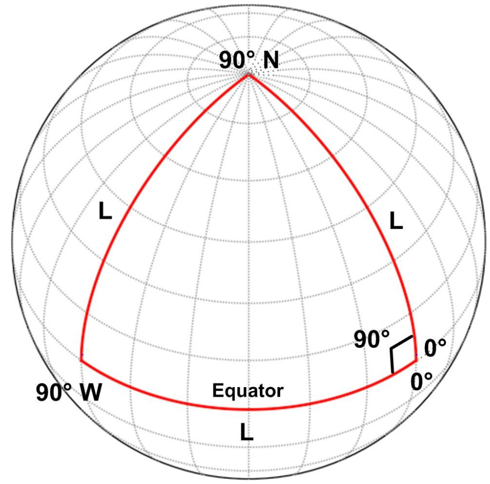

Figure 8 Octant of Sphere

Now the sum of the squares of the two legs is

$$\mathbf{L}^2 + \mathbf{L}^2 = 2\,\mathbf{L}^2$$

which would mean the hypotenuse should be $\sqrt{2}$ **L**. But the actual side is only of length **L**. Since $\sqrt{2} \approx 1.414$, the discrepancy is now 41%. So increasing the length of the legs from about 2000 miles to over three times as much at 6250 miles causes the hypotenuse excess to rise from 2% to 41%.

So we conclude that **if a surface satisfies the Pythagorean Theorem, it must be flat.**

Thus computing with the Pythagorean Theorem tells us something about the curvature of a surface. Why this is important is the way it tells us about curvature. If we never had airplanes or satellites to show us the spherical nature of the earth, all we could do is essentially move around on its surface to try to figure it out. Well, the measurements we used in the computations above could easily come from just such surface measurements, such as those made a century or two ago, before satellites and airplanes. Properties of surfaces that can be deduced solely from information on the surface are called **intrinsic**. Those properties that require the surface be embedded in a space of higher dimensions are called **extrinsic**.

Why is that significant? Well, we live in a 3-dimensional space, and we cannot determine the curvature of this space by looking at it embedded in a 4-dimensional space. The only way we can do this is intrinsically. Okay, what are the straight lines in 3-dimensional space? It turns out they are the pathways of beams of light. And what did Einstein predict in 1915 with his General Theory of Relativity, and what did Eddington observe in 1919? – that beams of light are bent as they pass near gravitational objects, such as the sun. This means gravitational objects bend our 3-dimensional space. (In fact, they bend the 4-dimensional spacetime, but that is for another time.) And so our space is curved. A very big question is the nature of this curvature. Is it like a sphere or like the surface of a saddle? Discussing the curvature of the 4-dimensional spacetime implies answers to the question of does the universe expand forever or does it stop and recontract?

Conclusion

So, far from being a useless, boring piece of trivia, the Pythagorean Theorem involves all kinds of fascinating mental gymnastics and plays a role in our probing and understanding of the universe around us. People who do not attempt to explore these ideas are condemned to wander in dark caves with limited vision and stunted lives.

Appendix: Another Geometric Proof

Among the many geometric proofs, there is one other that I find particularly appealing. It is based on an animated GIF found on the website http://www.mscroggs.co.uk/blog/27.

Area of a Parallelogram

The proof uses one additional property of areas that I should highlight, namely, that of the area of parallelograms (Figure 9).

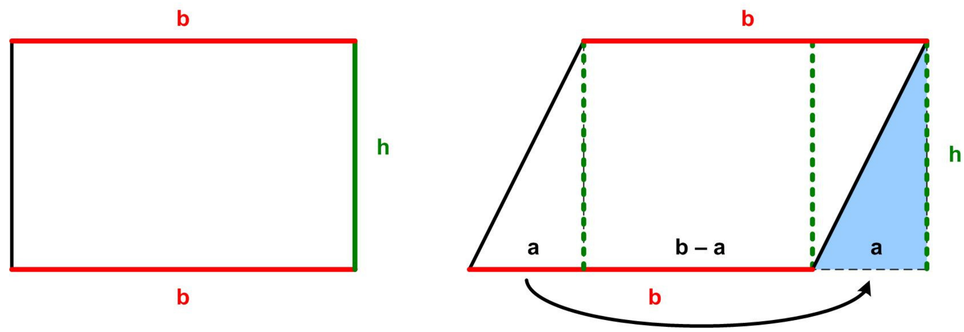

Figure 9 Area of a Parallelogram

Using the labels in the figure, we see the area of the parallelogram on the right is equal to the area of the rectangle on the left, namely bh. Visually this means we can take any rectangle and slide its upper side parallel to its lower side, always maintaining the same distance, and the resulting parallelogram will have the same area.

Rigorous Geometric Proof

We use this property for a nice, visual proof of the Pythagorean Theorem, based on the website http://www.mscroggs.co.uk/blog/27, as shown in Figure 10.

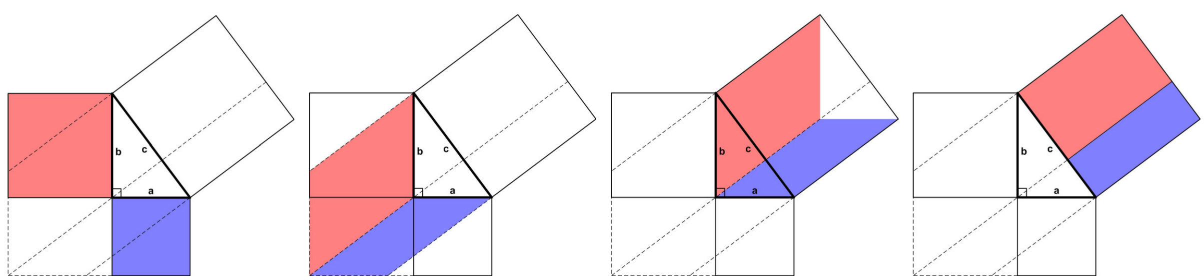

Figure 10 Alternative Geometric Proof of Pythagorean Theorem

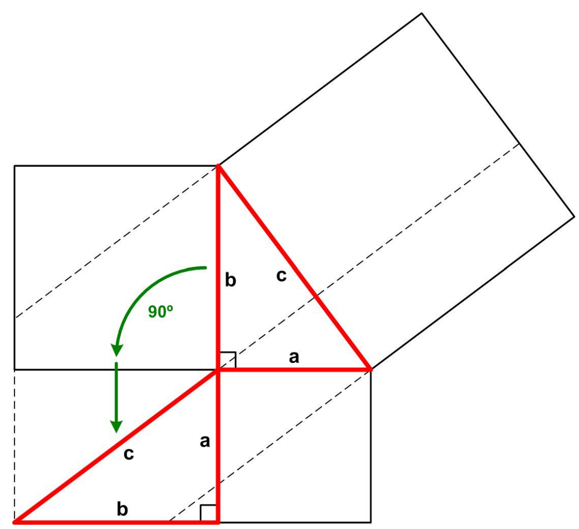

Figure 11 Congruent Triangles

There is only one step that might raise a question, namely from Figure 10(b) to Figure 10(c). That is, how do we know that the upper edge of the red parallelogram is exactly the length c of the side of the large square? This is equivalent to asking how do we know the diagonal of the rectangle on the lower left of the figure is the same as the edge of the large square? The answer is given in Figure 11.

The diagonal of the rectangle is also the hypotenuse of a right triangle that is the rotated (and translated) image of the original triangle. Therefore, this triangle is congruent to the original and has all the same lengths and angles. Thus the diagonal of the rectangle is also of length c, the length of the edge of the large square. And so this geometric manipulation really does constitute a rigorous proof of the Pythagorean Theorem.

Figure 12 gives the animated GIF of the proof as found on the website http://www.mscroggs.co.uk/blog/27.

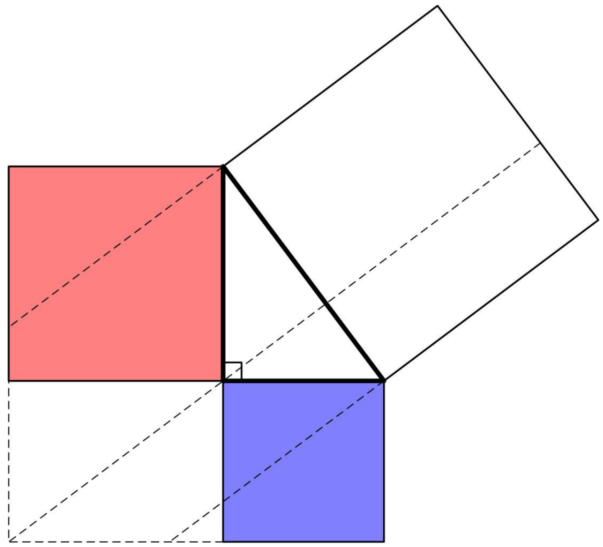

Figure 12 Animated GIF