Mercator Projection Balloon

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Years ago during one of my many excursions into the history of mathematics I wondered how Mercator used logarithms in his map projection (introduced in a 1569 map) when logarithms were not discovered by John Napier (1550-1617) and published in his book *Mirifici Logarithmorum Canonis Descriptio* until 1614, three years before his death in 1617.[1]

The mystery was solved when my father gave me a wonderful book on *The Art of Navigation in England in Elizabethan and Early Stuart Times* by D. W. Waters ([3]). There Waters explained that Edward Wright (1561-1615) in his 1599 book *Certaine Errors in Navigation* produced his "most important correction, his chart projection, now known as Mercator's"([3] p.220). Waters further said ([3] p.223):

> In introducing his new chart projection Wright forestalled accusations that he had stolen another man's work by admitting that it had been Mercator's well-known map of the world which had first prompted the idea of 'increasing the distance of the parallels, from the equator towards the Poles, so that at every point of latitude in the chart a part of the meridian had the same proportion to the same part of the parallel as in the globe'. He further avowed that it was neither from Mercator, nor any other man that he had learned 'the way how this should be done'.

Later Waters presents Wright's proof of the mathematics of the projection ([3] p.369n) and describes how Wright and others indicated the way to construct practical charts using the Mercator projection without involving logarithms, which we shall return to in a moment.

But first, attached to the above quote Waters had this arresting footnote: "Wright explained his projection in terms of a bladder blown up inside a cylinder, a very good analogy. See Pl. LX." (Figure 1).

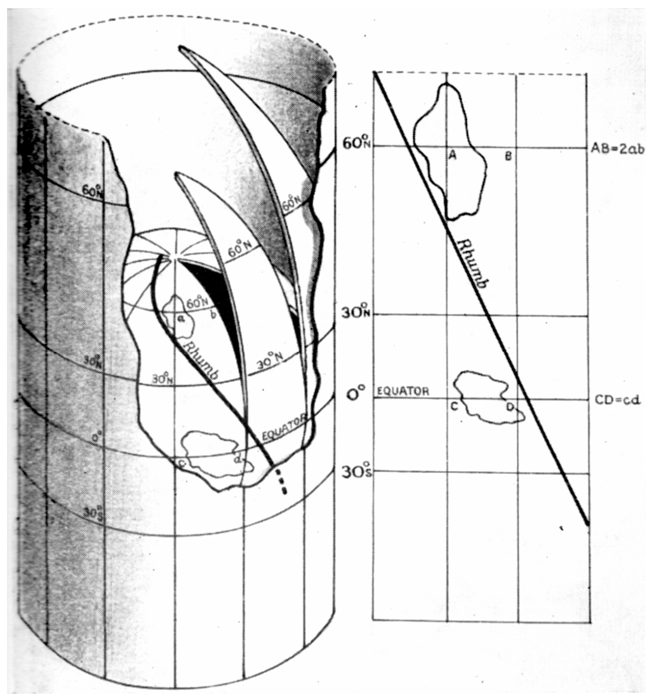

Figure 1 Plate LX from the *Art of Navigation* [p.232]

I remember taking a clear plastic sheet and wrapping it into a cylinder, and then taking a partially inflated balloon and drawing meridians and parallels on it to represent longitudes and latitudes. I then put the balloon inside the clear cylinder and inflated it. Sure enough the meridians showed up as straight vertical lines and the parallels as ever increasingly spaced horizontal lines perpendicular to the meridians. I had always wondered if this demonstration really showed the Mercator projection and if

[1] There are many sources of information on John Napier and his "natural" logarithms ($y = \ln x$), but the *Wikipedia* site is a good place to begin ([2]). Another recent source is Havil ([1]) The "common" logarithms, which are base 10 ($y = \log_{10} x$), are due to Henry Briggs (1561-1630) after conversations with Napier in 1615.

so, why. The present article is my attempt at explaining this.

Spherical Expansion

First, it is important to understand what happens to shapes (rectangles, triangles) drawn on a sphere as the sphere inflates (see Figure 2). “Straight” lines on a sphere are arcs of great circles, that is, circles which are the intersection of the sphere with a plane slicing through its center. Meridians and the equator are great circles, but parallels or latitude circles are not.

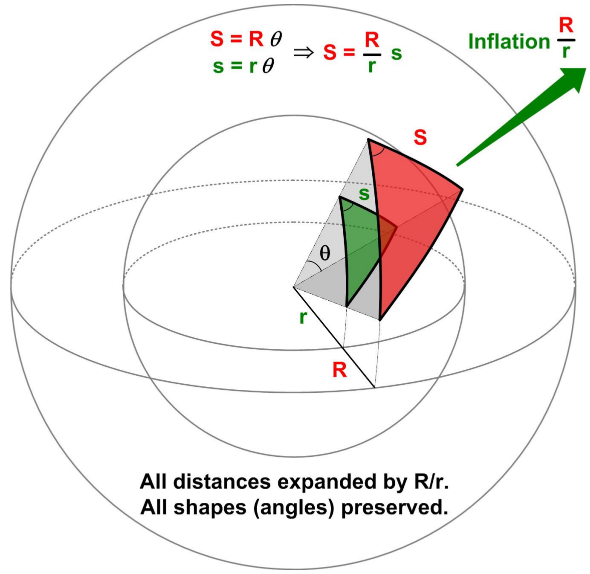

Figure 2 Inflating Balloon Properties

The lengths of such straight line segments are given as the length of the arc spanned by the angle measured from the center of the great circle, which is also the center of the sphere. If the angle is measured in radians rather than degrees, then the arclength = (radius of circle) x (angle in radians). The “distance” between two points on the sphere is measured by the length of the arc between them.

The angle between two arcs emanating from the same point on the sphere is given by the corresponding angle between the two intersecting planes slicing the sphere through its center to produce those arcs.

With these notions clearly in mind we can see that as the sphere expands uniformly in all directions, the angles between all intersecting lines remain the same, since the intersecting planes do not change. Thus shapes are preserved. And the distances (arclengths) all expand by the same ratio of the final sphere radius to the initial sphere radius.

Balloon in a Cylinder

Now we consider what happens when the expanding sphere is a balloon inside a cylinder (Figure 3). Consider a small “rectangular” patch on the balloon-sphere bounded by two latitude circles and two longitude circles (meridians). As the balloon expands, the region *above* the lower latitude circle inflates just as in Figure 2, equally in all directions. But the region *below* this latitude circle becomes constrained when it reaches the side of the cylinder and no longer expands. Consider the lower left-hand point in this lat-lon rectangle. The instant it has reached the wall of the cylinder, the entire patch has expanded by the ratio of the radius of the cylinder divided by the radius of the lower latitude circle. If we map this patch onto the cylinder using the lower expanded latitude arc and measuring the vertical height equal to the length of the expanded meridional segment between the two latitude circles, we get the desired projection patch on the cylinder.

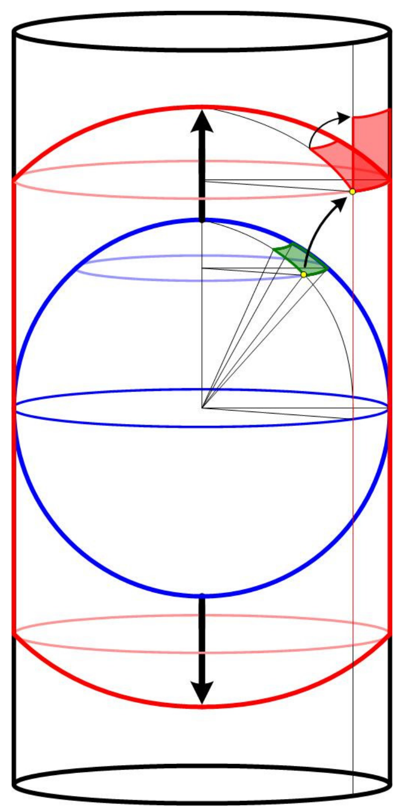

Figure 3 Balloon Inside Cylinder

Repeating this procedure from the equator to a sufficiently high latitude and around the equator produces a series of stacked sequences of small patches that together form the Mercator projected map, or at least a finite approximation to a smoothly varying final product as the increments between the latitudes and longitudes are shrunk to an infinitesimal size.

Mathematical Calculations

What are the exact calculations involved here that would prove we actually obtain the Mercator projection? We consider an incremental patch at a point on the sphere given by latitude λ and longitude θ (see Figure 4). The right-hand boundary will be longitude $\theta + \Delta\theta$ and the top boundary latitude $\lambda + \Delta\lambda$, where the Δ's represent small increments in value. The radius of the sphere is R and the radius of the latitude circle at λ is r. Therefore the arclength defining the lower edge of the (green) patch is $r\Delta\theta$ and the arclengths of the left and right sides of the patch are $R\Delta\lambda$. (The top edge is actually not defined by r but rather a smaller latitude radius. Nevertheless we shall approximate the patch by an exact rectangle using $r\Delta\theta$ for the top edge as well.)

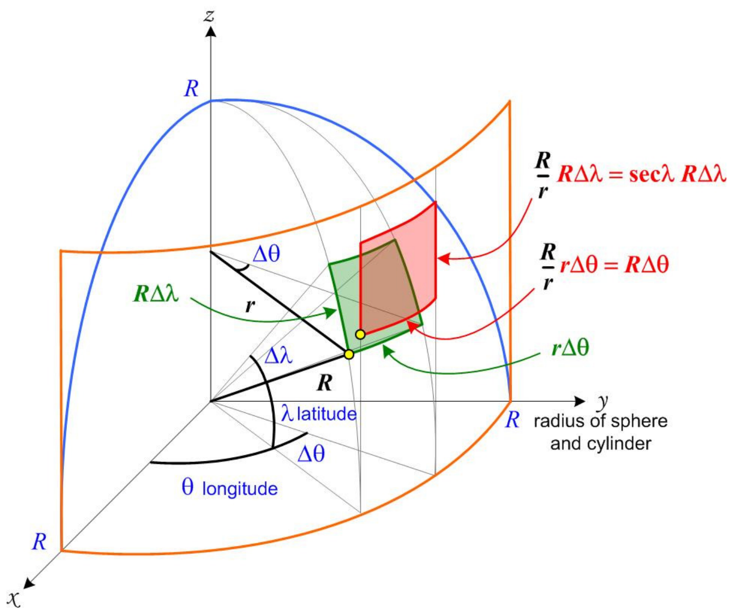

Figure 4 Computations for the Mercator Projection

From trigonometry we see that $r/R = \cos\lambda$ or $R/r = \sec\lambda$. This means that the (red) projected (R/r inflated balloon) patch has horizontal length $\Delta x = (R/r)(r\Delta\theta) = R\Delta\theta$ and vertical length $\Delta y = (R/r)(R\Delta\lambda) = (\sec\lambda)R\Delta\lambda$. (See Figure 5)

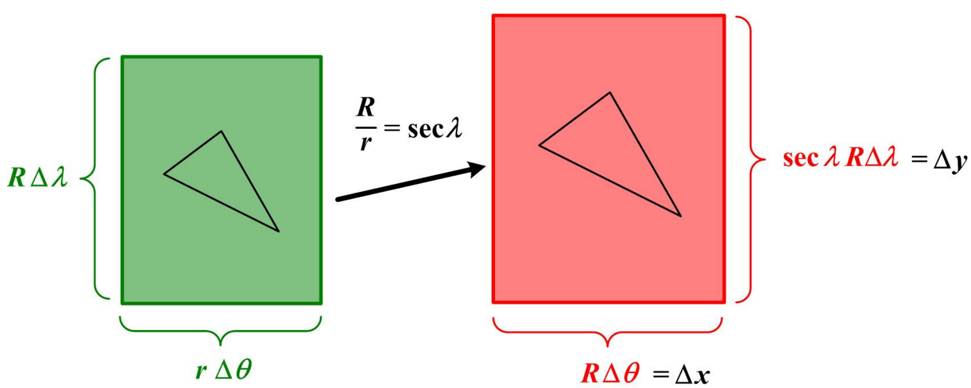

Figure 5 Patch Expansion Detail

So vertically stacking a sequence of Δx x Δy patches for a given longitude as we increment the latitude from the equator to some high latitude λ_{max}, via $\lambda_1 = 0, \lambda_2, \lambda_3, \ldots, \lambda_n = \lambda_{max}$, we have a strip of height

$$y = (\sec\lambda_1)R\Delta\lambda + (\sec\lambda_2)R\Delta\lambda + \ldots + (\sec\lambda_n)R\Delta\lambda$$

$$= R\,(\sec\lambda_1\,\Delta\lambda + \sec\lambda_2\,\Delta\lambda + \ldots + \sec\lambda_n\,\Delta\lambda) \quad (1)$$

This length, given in terms of secants, is the method employed by Wright to construct his charts. He computed a table of secants that could be used to compute the "meridional parts" that would be employed in the maps (see Waters [3] p.367).

Logarithms

So where is the logarithm? Well, as we take smaller and smaller increments in latitude $\Delta\lambda$, the sum in equation (1) approaches the limit of the integral

$$y = f(\lambda_{\max}) = R\int_0^{\lambda_{\max}} \sec\lambda d\lambda \tag{2}$$

For the purposes of the flat chart, we do not need to use the radius R (of the earth in this case), but rather any scale factor to convert the lengths on earth to inches on a sheet of paper.

To evaluate the integral in equation (2) we need to find a function whose derivative is sec λ. And it turns out this function is[2]

$$y = f(\lambda) = \ln(\tan\lambda + \sec\lambda)$$

and thus we finally have the appearance of the natural logarithm ($y = \ln x$) that defines the vertical distance y on the Mercator map. Of course the use of the calculus here at the time of 1600 is highly anachronistic, since its methods, and in particular the use of the Fundamental Theorem, were not made known until Newton and Leibniz presented them at the end of the century, some 90 years later.

Historical Note

We should not leave this discussion without mentioning that there eventually was a close tie between Edward Wright and John Napier's logarithms. Waters tells the story best ([3] pp.402-405):

> IT was in 1614 that John Napier, laird of Merchiston, a property then on the outskirts of Edinburgh, published in that city a small quarto volume of one hundred and forty-seven pages entitled *Mirifici Logarithmorum Canonis Descriptio.* It consisted of ninety pages of mathematical tables and fifty-seven pages of explanatory text written, as became a work intended for scholars in all lands, in Latin. Probably no work has ever influenced science as a whole, and mathematics in particular, so profoundly as this modest little book. It opened the way for the abolition, once and for all, of the infinitely laborious, nay, nightmarish, processes of long division and multiplication, of finding the power and the root of numbers, that had hitherto been the inescapable lot of every mathematician in every walk of life. It described and tabulated Napier's invention of logaritbms—'the rare and exquisite *Inuention* of the Logarithmes' as it was soon called—and 'gaue directions how to resolue all the *Propositions* of *Trigonometrie* by *Addition,* and *Subtraction,* which were never performed before without *Multiplication,* and *Division* …'. …
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> Meanwhile, Wright, like Briggs, had also perceived the importance of Napier's work. Moreover, with his strong navigational bent he had seen that, if the Latin text was put into plain English, it would prove to be 'of very great use for Mariners ... a booke of more than ordinary worth, especially for Sea-men'. Accordingly, with the encouragement and, through his lectureship, the financial support of the East India Company, he had undertaken the task of translation. He had submitted the result, together with a diagram for finding proportional parts which he had devised to simplify interpolation, to Napier for his approval. This Napier conceded, but unfortunately Wright died before he could complete the work, apparently in December 1615.
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> On the news of Wright's death the indefatigable Briggs immediately undertook to complete his work and, with Wright's son Samuel, to see it through the press. It appeared the next year, *A Description of the Admirable Table of Logarithmes.*

[2] $f'(\lambda) = (1/(\tan\lambda + \sec\lambda))(\sec^2\lambda + \sec\lambda\tan\lambda) = \sec\lambda$

… Thus, and it is probably not generally realized today, logarithms were first brought into popular use primarily in the interests of easier and more accurate navigation.

References

1. Havil, Julian, *John Napier: Life, Logarithms, and Legacy*, Princeton University Press, New Jersey.2014.

   This recent book discusses Napier's approach in excruciating detail. What makes it so difficult is that it is from the vantage point of the late Renaissance when the use of mathematical notation and symbols was just in its infancy. Moreover, Napier was approaching the subject as a means of solving computational problems in astronomy and navigation, that is, spherical geometry, so his initial presentation was constrained by that perspective. After reading what Napier had to overcome to arrive at the notion of logarithms, one appreciates the simplicity, clarity, and generality of the modern formulations.

2. "John Napier." *Wikipedia* (https://en.wikipedia.org/wiki/John_Napier), retrieved 3/4/2017.

3. Waters, David Watkins, *The Art of Navigation in England in Elizabethan and Early Stuart Times*, Yale University Press, New Haven, 1958. A searchable PDF version can be found online at https://babel.hathitrust.org/cgi/pt?id=uc1.b4522194;view=1up;seq=496 (retrieved 3/2/2017). A non-searchable PDF version is at https://archive.org/details/in.ernet.dli.2015.51856 (retrieved 3/2/2017).