Lambert Equal-Area Cylindrical Map Projection

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One thing I have always been curious about, but never got around to investigating, is how hard is it to see that the Lambert Equal-Area Projection of a sphere onto a cylinder in fact preserves areas?

Figure 1 depicts the Lambert Equal-Area Cylindrical Projection where each non-polar point on the sphere is mapped to a point on the cylinder wrapped around its equator by drawing a line from the axis of the sphere (perpendicular to the plane of the equator) horizontally through the given point on the sphere until the line intersects the cylinder. That point of intersection is the projected point. The cylinder is then cut along a meridian and rolled out flat to give the map.

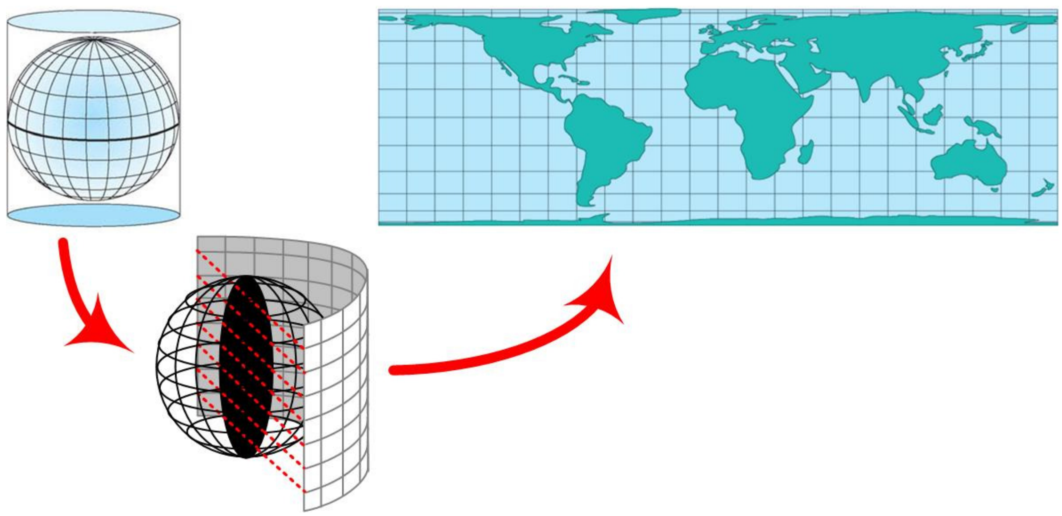


Figure 1 Lambert Equal-Area Cylindrical Projection

The Lambert projection is an equal-area mapping if the area of any region on the sphere has the same value as the area of its projected image on the cylinder (see Figure 2). Because all the surfaces involved are curved, we have to resort to calculus. Rather than carry out detailed calculations, we will suggest intuitively how this is done.

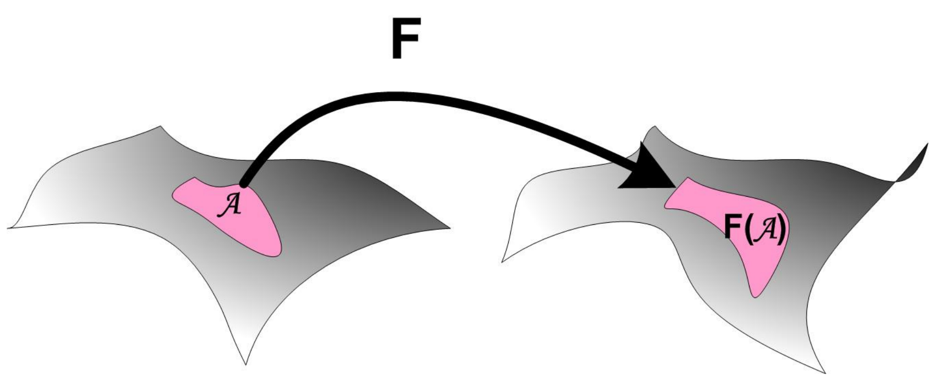
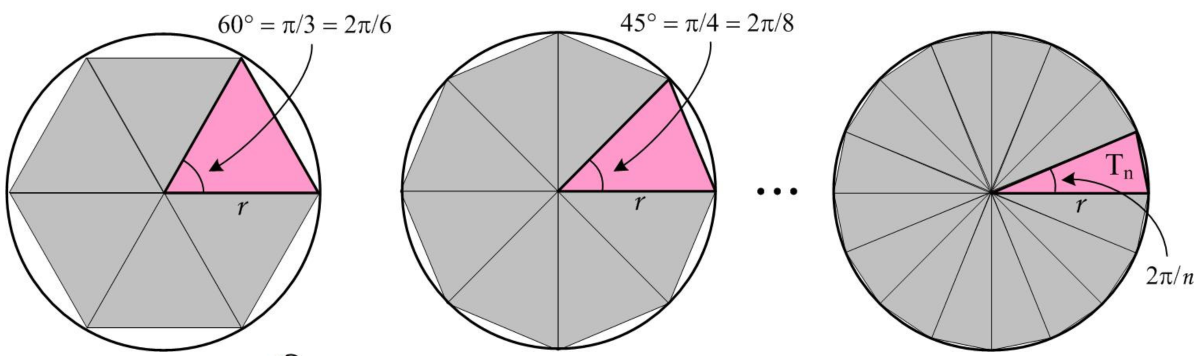


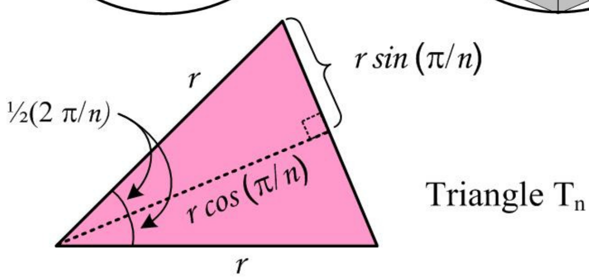
Figure 2 F is an equal-area mapping if F($\mathcal{A}$) = $\mathcal{A}$ for any area in its domain

In calculus the area of a region is obtained by *integration*. This process amounts to covering the region of interest with a succession of sets of contiguous tiles, where each set consists of an increasing number of smaller and smaller tiles which approximate the region better and better. The area is taken to be the limit of the areas of this succession of covering sets. The idea is to select the tiles so that their individual areas are easy to compute and we can just add them up to get the area of the whole set.

Example: Area of Circle

These ideas are illustrated in Figure 3, following the example of Archimedes, the famous third century BC Greek mathematician, who calculated the area of a circle by finding the areas of successive regular polygons of *n* sides inscribed in the circle. As the number of sides *n* grew, the polygon P_n approximates the circle better and better, and so the area of the circle is the limit as *n* grows without bound of the areas of P_n. The area of the nth polygon P_n is computed by subdividing the polygon into *n* equal triangles T_n ("tiles") as shown in Figure 3. Then the area of P_n is $n\,T_n$.





h successive regular polygons with triangular tiles.

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Recall that the area of a triangle is one half the base times the altitude. Using Figure 4, we see the area of T_n is

Figure 4 Calculating Area of Triangle T_n

$$T_n = r\sin\left(\frac{\pi}{n}\right) r\cos\left(\frac{\pi}{n}\right)$$

$$= \tfrac{1}{2} r^2\, 2\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)$$

$$= \tfrac{1}{2} r^2 \sin\left(2\frac{\pi}{n}\right) \quad \text{(trigonometry identity)}$$

Therefore, we have for the area of polygon P_n,

$$P_n = nT_n = \frac{n}{2} r^2 \sin\left(\frac{2\pi}{n}\right) = \pi\frac{n}{2\pi} r^2 \sin\left(\frac{2\pi}{n}\right) = \pi r^2 \frac{\sin x_n}{x_n}$$

where $x_n = 2\pi / n$. So as *n* becomes large (written $n \to \infty$), $x_n \to 0$ and $\sin x_n / x_n \to 1$ (proved in a calculus course), and so $P_n \to \pi r^2$, the area of a circle of radius r.

Back to the Lambert Cylindrical Projection. The way we will show it is equal-area is by showing it projects each tile in a covering of a region of the sphere onto a tile of equal area on the cylinder. There will be a little fudging (approximations), but these differences will vanish in the limit. We

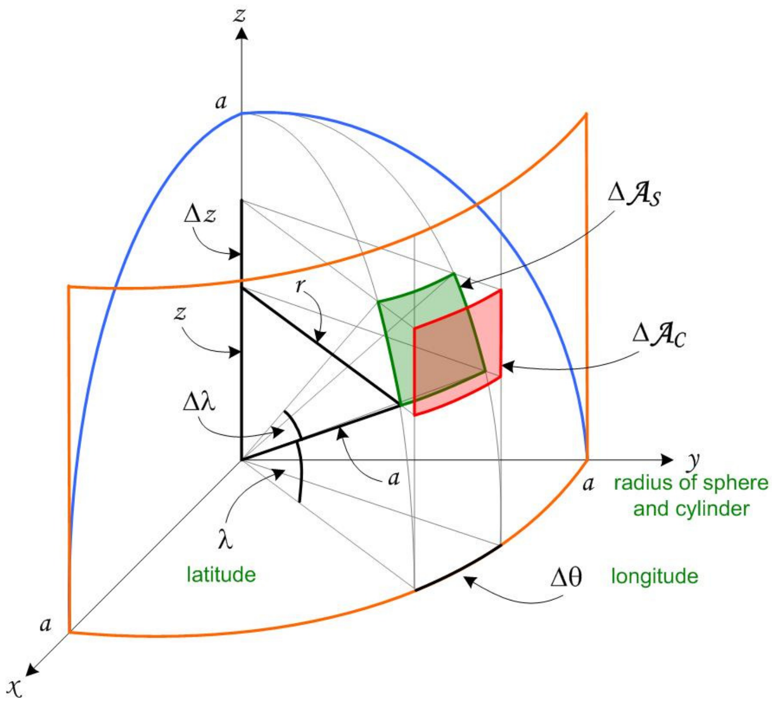

Figure 5 Lambert projection of area patch $\Delta\mathcal{A}_S$ onto $\Delta\mathcal{A}_C$

choose the tiles or patches on the sphere to be little curvilinear "rectangles" bounded by latitude and longitude circles (equivalently parallels and meridians). Figure 5 shows an example of such a patch (in green) and its projection onto the cylinder (in red).

We now have the issue of how to measure the area of these patches. The cylindrical patch is not a problem, since we can slice the cylinder along a meridian and lay it down flat. This distance-preserving motion does not change the area of the patch, so it becomes a simple rectangle of height Δz and width $a\,\Delta\theta$, where $\Delta\theta$ is the change in longitude measured in radians and a is the radius of the cylinder.

The green patch on the sphere is more of a problem. A most remarkable theorem of Gauss (basically) states that any distance-preserving transformation of one surface into another must also preserve its intrinsic curvature. There are technical issues in the definition of curvature here that I will skip (since a curved sheet like a cylinder can be laid flat on a plane (preserving distances) and the plane's curvature is clearly zero, it must mean the cylinder's intrinsic curvature is also zero as well!). But the sphere's intrinsic curvature is $1/a^2 > 0$ where a is the radius of the sphere. So we are going to have to resort to some approximations to represent the patch on a flat plane in order to measure the area. One approach would be to pick the lower left corner of the patch and lay out a rectangle using the lengths of the arcs leading from this corner, that is, a rectangle with height $a\,\Delta\lambda$ and width $r\,\Delta\theta$. Since the meridians are slanting inward toward the poles, the patch is really more like a trapezoid with the upper edge shorter than the lower edge (r shrinks as we move northward). But this difference from our rectangle (as well as the curvature of the right-hand edge) diminishes as we take smaller and smaller patches. (See Figure 6)

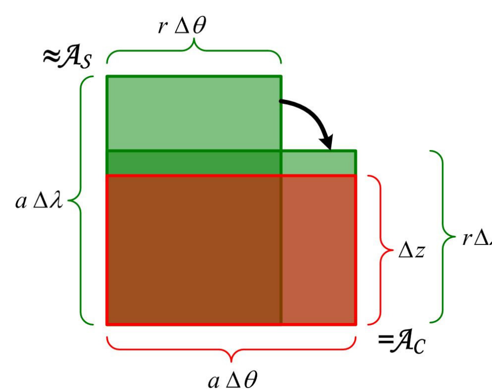

Figure 6 Rectangular Area Approximations

So as a first step we have for the areas of the cylindrical patch and spherical patch respectively:

$$\mathcal{A}_C = (\Delta z)(\,a\,\Delta\theta\,) \quad \text{and} \quad \mathcal{A}_S \approx (\,a\,\Delta\lambda\,)(\,r\,\Delta\theta\,) = (\,r\,\Delta\lambda\,)(\,a\,\Delta\theta\,) \tag{1}$$

In order to show these two areas converge to one another in the limit as we shrink the latitude and longitude spacing, we only need to show $r\,\Delta\lambda \to \Delta z$ (see Figure 6). From Figure 5 we see that $z = a \sin\lambda$. We need a way to relate Δz and $\Delta\lambda$. We have

$$\begin{aligned}\Delta z &= (z+\Delta z) - z = a\sin(\lambda+\Delta\lambda) - a\sin\lambda \\ &= a[(\sin\lambda\cos\Delta\lambda + \cos\lambda\sin\Delta\lambda) - \sin\lambda] \qquad \text{(trigonometry identity)} \\ &= a[\sin\lambda(\cos\Delta\lambda - 1) + \cos\lambda\sin\Delta\lambda]\end{aligned} \tag{2}$$

We need to consider some further approximations as $\Delta\lambda \to 0$. (Notice we don't have to worry about $\Delta\theta \to 0$.) From the properties of the cosine we know that for small angles the cosine is almost 1, so $\cos\Delta\lambda \to 1$, as $\Delta\lambda \to 0$, and so the first term in equation (2) becomes negligible. From the limit we mentioned explaining the area of the circle, namely, $\sin x_n / x_n \to 1$, as $x_n \to 0$, we have $\sin\Delta\lambda \to \Delta\lambda$, as $\Delta\lambda \to 0$. Putting all this together with equation (2) yields (see Figure 5)

$$\Delta z \approx (a\cos\lambda)\,\Delta\lambda = r\,\Delta\lambda \tag{3}$$

which is what we wanted to show. And so $\mathcal{A}_S \to \mathcal{A}_C$ as we take smaller and smaller patches on the sphere, and that implies that the areas of regions of the sphere are preserved under the Lambert Cylindrical Projection.

(**Update 12/4/2018**) A terrific Youtube video that includes a more intuitive and visual explanation of the idea of the equal area projection is "But WHY is a sphere's surface area four times its shadow?" (12/2/2018) (https://www.youtube.com/watch?v=GNcFjFmqEc8) by Grant Sanderson at 3blue1brown.