Measure of a Degree of Latitude and the Equatorial Bulge

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Jim Stevenson

The following are some thoughts after reading Michael Dirda's review in the Sunday *Washington Post* of Ferreiro's *Measure of the Earth* [1]. The book described the 1735 Geodesic Mission, whose purpose was to resolve the question of the shape of the earth, that is, whether it was a sphere, or like an egg with the poles further from the center than the equator, or like an oblate spheroid with the equator further from the center than the poles, as Newton averred due to centrifugal force. In the review Dirda said, "A team, sympathetic to Newton's view, would travel to what is now Ecuador and measure the exact length of a degree of latitude near the equator. This would then be compared with the same measurement taken in France. If the latter was larger, Newton was right."

Now I thought that was backwards, that a degree at the equator would be larger than one at a higher latitude if the earth bulged. But then I realized a more serious question is what is a degree of latitude, if you no longer assume the earth is a sphere? The following provides some of the late USGS cartographer John P. Snyder's definitions (Figure 1).

Latitudes (From John P. Snyder's book [2]:)

- **geographic or geodetic latitude φ** ([2] p. 13): angle which a line perpendicular to the surface of the ellipsoid at the given point makes with the plane of the Equator (which is normally the latitude referred to for a point on the Earth).
- **geocentric latitude φ_g** ([2] pp. 13, 17): the angle made by a line from the given point to the center of the ellipsoid with the equatorial plane.

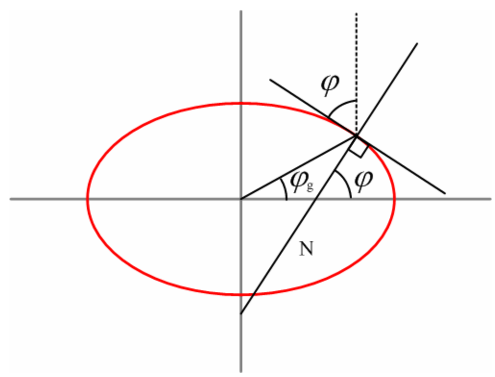

Figure 1 Latitude Definitions

Geocentric latitude. I can see I was mentally imagining latitude was measured from the center of the earth (geocentric latitude), but a surveyor would not know where that is. So it looks like the more likely definition is equivalently the angle made by the north

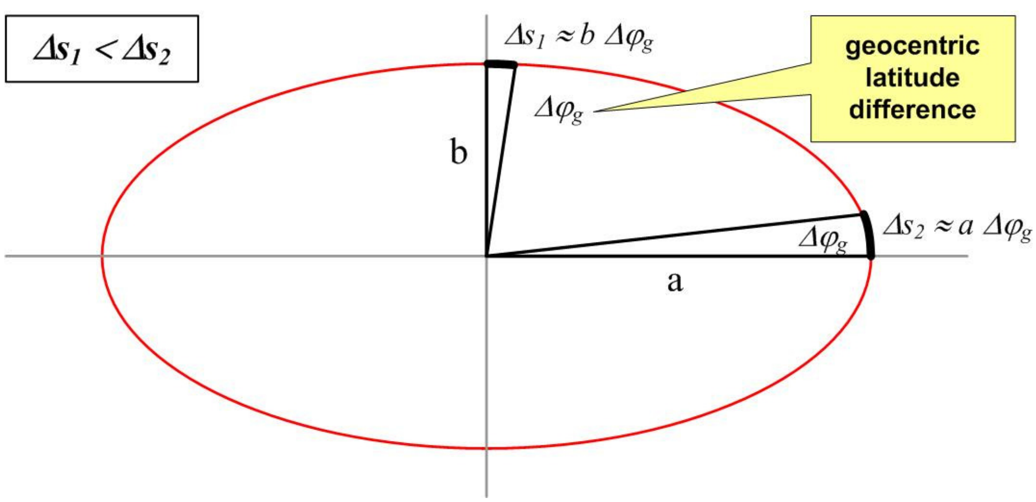

Figure 2 Arc Length Comparison for Geocentric Latitude Definition

polar axis with the horizon (geographic or geodetic latitude). The question of how such a latitude was measured at different points in history is a tale for another time. In any case Figure 2 shows what I was thinking and sure enough, for the same difference in latitude $\Delta\varphi_g$, the arc length at the equator Δs_2 is greater than that at the pole Δs_1 (since the semi-major axis a of the ellipse is greater than the semi-minor axis b).

Geographic (geodetic) latitude. So how are we to understand the effect of measuring the geodetic latitude? For a sphere, the geocentric and geodetic latitudes are the same, since the radius of the sphere is also the normal to its surface. To measure the length of a meridian segment on an oblate spheroid "earth" (rotationally symmetric ellipsoid, that is, a surface generated by an ellipse rotated about its minor axis), take the intersection of a plane slicing the ellipsoid through its axis of rotation along the meridian of interest, thus producing an elliptical cross-section. Again consider a point at

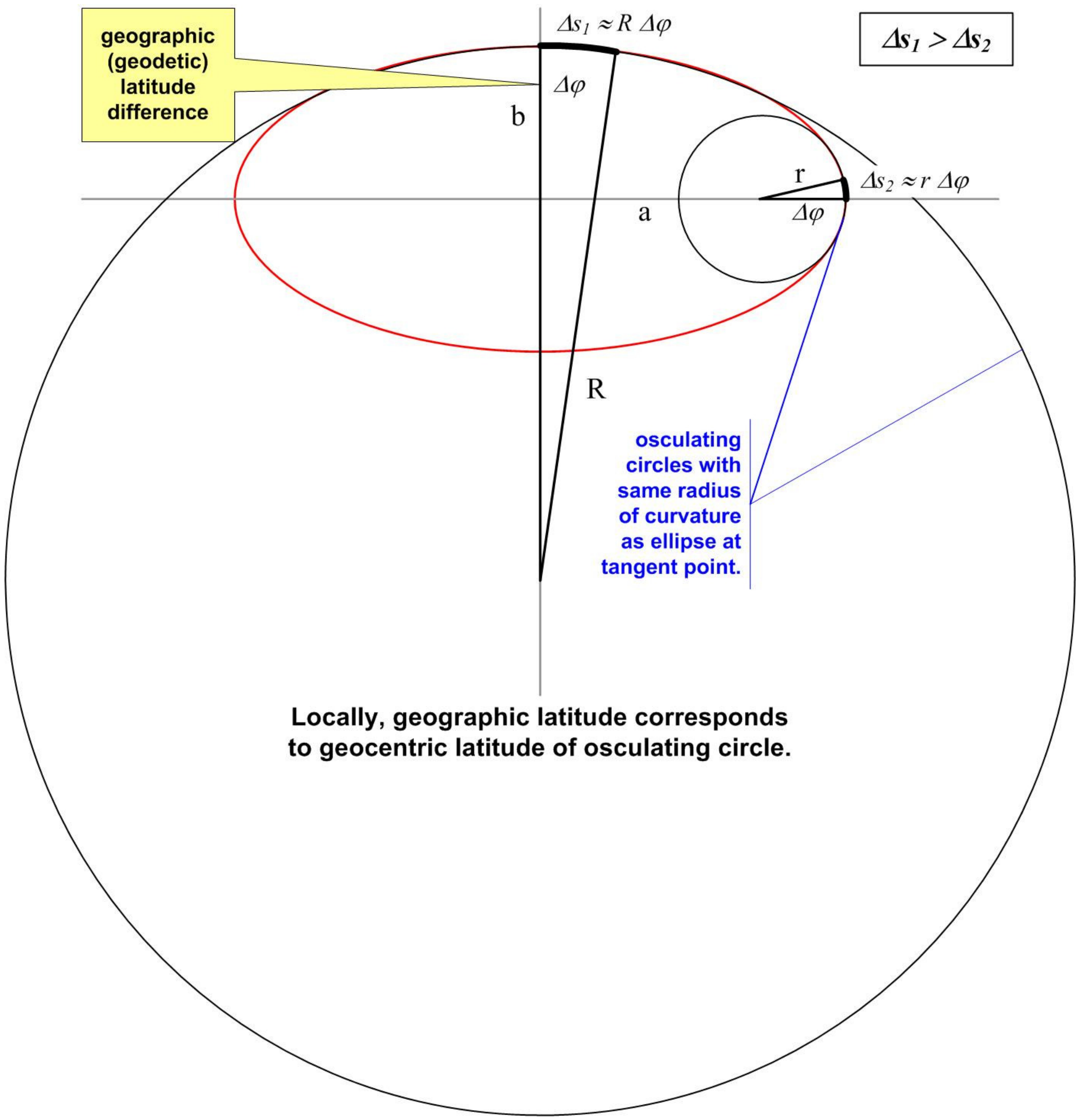

Figure 3 Arc Length Comparison for Geographic Latitude Definition

the equator and a point at the north pole. Find the two osculating ("kissing") circles to the ellipse at these two points (see Figure 3). These circles have the same tangents as the ellipse at the respective points, and also have the same curvature, and hence, radii of curvature (recall, *radius of curvature = 1 / curvature*, and also recall the radius of curvature of a circle is its radius). It should be clear that a close approximation to the elliptical geodetic latitude in the neighborhood of the tangent points would be the geodetic (equivalently, geocentric) latitude of the osculating circle. And since the curvature of the ellipse at the pole is smaller than at the equator, the radius of curvature R of the polar osculating circle is larger than the radius of curvature r of the equatorial osculating circle. So for the same latitude difference $\Delta\varphi$ the corresponding arc length at the pole Δs_1 is larger than that at the equator Δs_2.

It should be clear that if an intermediate point is considered instead of one at the pole, the argument still holds, since the radius of curvature increases as we move along the elliptical meridian from equator to pole.

References

[1] Dirda, Michael, "Larrie D. Ferreiro's 'Measure of the Earth,'" in *Washington Post*, Sunday, 31 July 2011 (http://www.washingtonpost.com/entertainment/books/larrie-d-ferreiros-measure-of-the-earth-reviewed-by-michael-dirda/2011/07/25/gIQAaG9adI_story.html)

[2] *Map Projections – A Working Manual*, John P. Snyder, U.S. Geological Survey Professional Paper 1395, US Government Printing Office, 1987