Astronomical Sum

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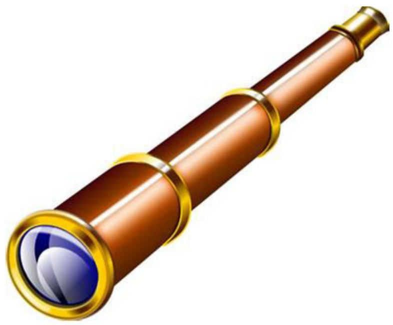

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This is a problem from the 2000 Olymon (the Mathematics Olympiads Correspondence Program) for secondary students sponsored jointly by the Canadian Mathematical Society and the Mathematics Department of the University of Toronto.

Let

$$S = \frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \frac{3^2}{5\cdot 7} + \ldots + \frac{500^2}{999\cdot 1001}.$$

Find the value of S.

My Solution

This is quasi-telescoping sum (and thus the pun). The kth term in the sum satisfies

$$\frac{k^2}{(2k-1)(2k+1)} = \frac{1}{4}\left(\frac{k}{2k-1} + \frac{k}{2k+1}\right).$$

Therefore,

$$4S = \left(\frac{1}{1}+\frac{1}{3}\right) + \left(\frac{2}{3}+\frac{2}{5}\right) + \left(\frac{3}{5}+\frac{3}{7}\right) + \ldots + \left(\frac{499}{997}+\frac{499}{999}\right) + \left(\frac{500}{999}+\frac{500}{1001}\right)$$

$$= \frac{1}{1} + \left(\frac{1}{3}+\frac{2}{3}\right) + \left(\frac{2}{5}+\frac{3}{5}\right) + \left(\frac{3}{7}+\frac{4}{7}\right) + \ldots + \left(\frac{499}{999}+\frac{500}{999}\right) + \frac{500}{1001}$$

$$= 1+1+1+1+\ldots+1+\frac{500}{1001} = 500 + \frac{500}{1001} = 500\left(\frac{1002}{1001}\right)$$

from which

$$\boxed{S = 125\left(\frac{1002}{1001}\right) = \frac{125250}{1001}}$$

So the sum is not quite telescoping in the sense of collapsing with the cancellation of the intermediate terms, but it does involve expanding a product into a sum which is then easy to add up from simplified intermediate terms.

Olymon Solution

The first Olymon solution is the same as mine. The second is more or less similar.

Solution 2. [Samer Seraj] We have that

$$\begin{aligned}
\sum_{i=1}^{n} \frac{i^2}{(2i-1)(2i+1)} &= \frac{1}{2}\sum_{i=1}^{n}\left[\frac{i^2}{2i-1} - \frac{i^2}{2i+1}\right] \\
&= \frac{1}{2}\left[1 + \sum_{i=1}^{n-1}\left(-\frac{i^2}{2i+1} + \frac{(i+1)^2}{2(i+1)-1}\right) - \frac{n^2}{2n+1}\right] \\
&= \frac{1}{2}\left[1 + \sum_{i=1}^{n-1} 1 - \frac{n^2}{2n+1}\right] = \frac{1}{2}\left[n - \frac{n^2}{2n+1}\right] = \frac{n(n+1)}{2(2n+1}\ ,
\end{aligned}$$

When $n = 500$, we get the answer $125250/1001$.