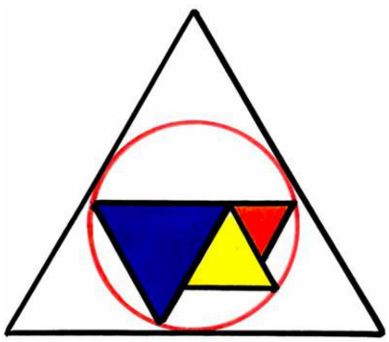
Four Equilateral Triangles

23 July 2025

Jim Stevenson



This is an old puzzle from Catriona Agg that I found on BL's Math Games website.[1]

All four triangles are equilateral. What fraction of the largest triangle is shaded?

This problem turned out to be both easy and unexpectedly challenging, at least for me.

My Solution

Figure 1 shows the constraints in the problem. The vertices of the three colored triangles signified by the colored dots lie on the circle inscribed in the large triangle. A vertex from each of the three colored triangles coincides in the white dot. Finally, the top edges of the blue and red triangles form a line parallel to the base of the large triangle.

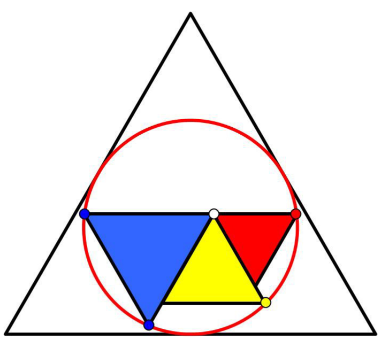


Figure 1

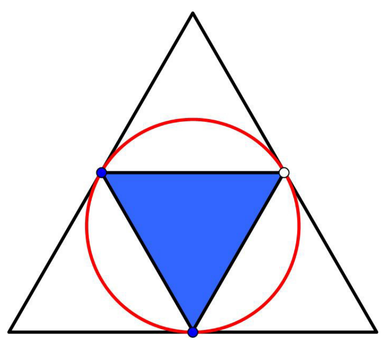


Figure 2

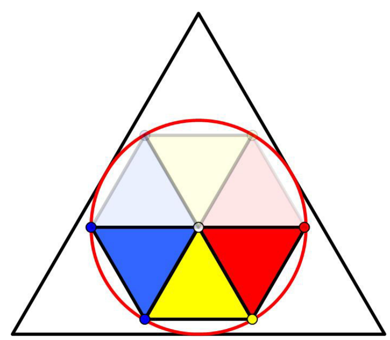


Figure 3

These constraints allow for a multitude of configurations of the three colored triangles, ranging from the case where the areas of the yellow and red triangles approach zero in the limit (Figure 2) to the case where all the colored triangles are congruent (Figure 3). Notice that the requirement that two of the blue triangle's vertices lie on the circle means its edge can't be smaller than the radius of the circle. If that were the case, we would have an example that is like Figure 1, only flipped horizontally about the vertical altitude of the large triangle.

Therefore, if the problem is well-defined, all configurations should yield the same answer. So following the Polya principle, chose a configuration that is easy to compute: one of the extremes. The simplest one is in Figure 2, and we see instantly that the area is ¼ the area of the large triangle.[2] So we easily have our answer.

Just as a test, we should compute the area of the other extreme in Figure 3. To do this we first need to compute the areas in Figure 2 (Figure 4).

[1] 17 July 2025 (https://medium.com/math-games/tell-me-why-meta-reasoning-works-this-time-337ca4ef02ee)

[2] Admittedly, some more explicit reasoning is required to show the blue small triangle is congruent to the other three small triangles (sides of parallelograms, etc.).

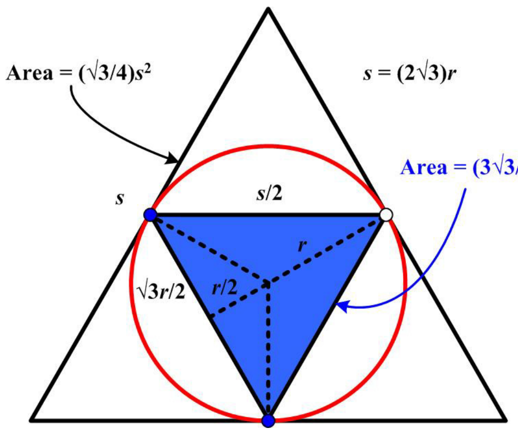


Figure 4

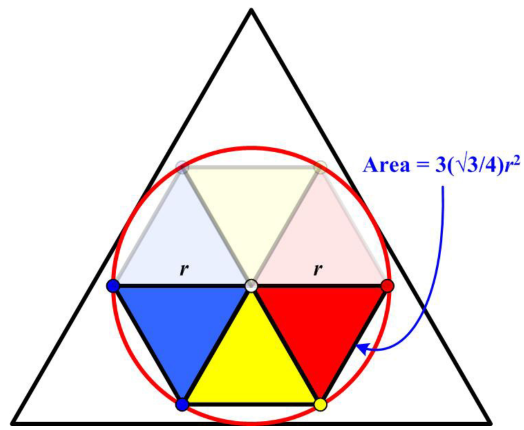


Figure 5

Let s be the length of the side of the large triangle and r the length of the radius of the inscribed circle. Then because the triangles are all equilateral we have the area of the large triangle is $(\sqrt{3}/4)s^2$ and that of the blue triangle $3(\sqrt{3}/4)r^2$. We also see from the diagram that $s = (2\sqrt{3})r$. Therefore,

$$(\sqrt{3}/4)s^2 = (\sqrt{3}/4)12r^2 = 4(3\sqrt{3}/4)r^2$$

which confirms computationally that the area of the blue triangle is ¼ that of the large triangle.

In Figure 5 the area of one of the smaller triangles is $(\sqrt{3}/4)r^2$, and we have three of them. So the area of the shaded region is $3(\sqrt{3}/4)r^2$, which, as we just saw, is ¼ the area of the large triangle.

So now we turn to the problem of showing the sum of areas of the three triangles in any allowable configuration is constant. But first we need the following.

Claim. *The altitude of the blue triangle is collinear with the altitude of the large triangle.*

While constructing various configurations for the problem in Visio, I noticed that the common vertex (white dot) was moving along an altitude of the large triangle (Figure 6).

To prove this, notice that the top edge of the blue triangle being parallel to the bottom edge of the large triangle, means its left edge is parallel to the right edge of the large triangle, and so perpendicular to the altitude. So the blue triangle's altitude is parallel to the large triangle's altitude.

The perpendicular bisector of a chord on a circle passes through its center. But the altitude of the large triangle also passes through the center of the circle. So the blue triangle altitude extended and the large triangle altitude have a point in common, and so are collinear.

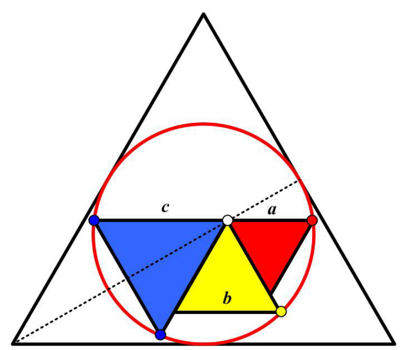


Figure 6

Therefore, the white dot is moving along a diameter of the circle. Now for the constant sum of areas.

Claim. *The sum of the areas of the colored triangles is constant.*

Since I am not subscribed to Twitter or Instagram, I could not see how others solved Agg's original post of the problem. And not subscribing to BL's website, I could not see his full solution, but more about that below. Given that it was a Catriona Agg puzzle I thought there should be a nifty, simple solution to show the sum of the areas were constant, but I couldn't find it. So after a lot of trial and error I arrived at the following (complicated) argument.

I needed some way to tie the colored triangles to the circle, in particular to the constant radius r. Since the white dot is moving along a diameter, I extracted the sides of the colored triangles, a, b, and c, and isolated them with the radii of the circle (Figure 7). The variable x is the distance between the common vertices of the triangles (white dot) and the center of the circle. As x ranges from 0 to r, the configuration covers all possibilities from that in Figure 3 to the one in Figure 2. Hopefully we can find an expression that is independent of x.

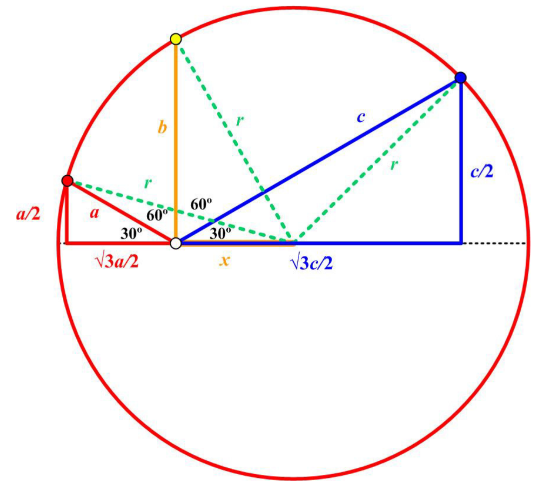


Figure 7

Notice that the areas of all the equilateral triangles are a constant multiple (√3/4) of the square of their sides (see the numerical computations above). Therefore it suffices to show the sum of squares of the sides of the triangles is constant, that is, that

$$a^2 + b^2 + c^2 = \text{constant (only depending on } r\text{)}.$$

Three applications of the Pythagorean Theorem yield

$$r^2 = \left(\frac{a}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}a + x\right)^2 = a^2 + \sqrt{3}ax + x^2 \quad (1)$$

$$r^2 = b^2 + x^2 \quad (2)$$

$$r^2 = \left(\frac{c}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}c - x\right)^2 = c^2 - \sqrt{3}cx + x^2 \quad (3)$$

Adding equations (1) – (3) yields

$$3r^2 = a^2 + b^2 + c^2 - \sqrt{3}\, x(c - a) + 3\, x^2 \quad (4)$$

Subtracting equation (1) from equation (3) produces

$$c^2 - a^2 = \sqrt{3}\, x\, (c + a),$$

where $c \geq r > 0$ implies $c + a > 0$, and so

$$c - a = \sqrt{3}\, x. \quad (5)$$

Applying equation (5) to equation (4) finally gives us

$$a^2 + b^2 + c^2 = 3r^2 \quad (6)$$

which only depends on r and not x. Done.

Notice that if we multiply equation (6) by our constant factor √3/4, we get an equation for the areas, where the right side is the area of the blue triangle in the extreme case in Figure 4,

$$(\sqrt{3}/4)a^2 + (\sqrt{3}/4)b^2 + (\sqrt{3}/4)c^2 = 3(\sqrt{3}/4)r^2$$

(How in the world did Catriona Agg ever think of this problem? Is there a simple plane geometry proof?)

BL Solution

With his discussion of a "meta puzzle" BL seems to be indicating the Polya principle we exercised above in looking at a simpler case to yield the answer. He also seems to be challenging the solver to show the sum of areas is constant for the allowable configurations, so that the problem is well-defined.

Thanks to a fellow reader of Math Games Jonathan Lynn Harvey,[3] I learnt that a 'meta' puzzle is a puzzle about a puzzle, akin to a metapoem being a poem about poetry per se.

In the context of geometry puzzles, a meta puzzle would involve finding out the hidden tricks within the setup of any given figure. More often than not, as many readers have pointed out in the last challenge, varying certain dimensions of the figures have no impact on the answer. And it's through this one can arrive at the answer faster than the conventional way.

Of course, today's puzzle is one of those meta puzzles. In fact, there are multiple configurations of how the 3 middle equilateral triangles can be positioned. This would make the solution way more elegant, albeit not mathematically rigorous.

So my challenge for you is not only to find the answer, but also proves to me concretely why the meta logic works for this specific puzzle.

Now without further ado, let's dive in.

Solution

Again, I don't subscribe to BL's website, so I can only surmise what his solution might be from the one diagram (Figure 8) he provided before the rest was hidden behind his subscription wall.

BL is using the parameter a to generate all the allowable configurations, where a is the distance from the center of the circle to the horizontal line of the upper edges of the blue and red triangles. The parameter a varies from 0 to ½, or $r/2$ for a radius of $r = 1$.

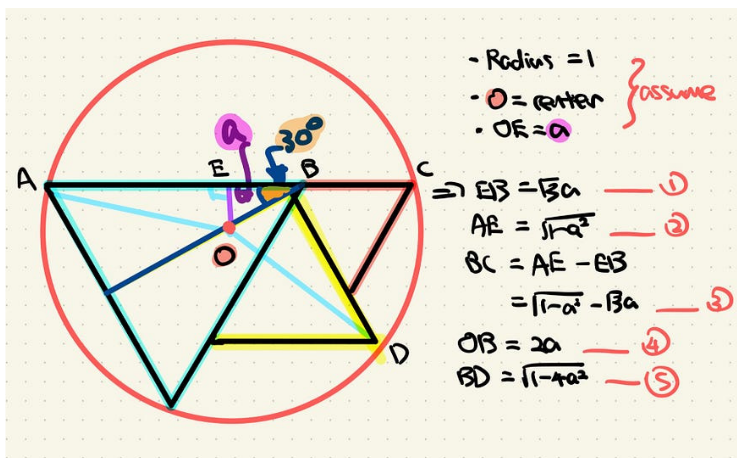


Figure 8

Figure 8 shows that BL has computed the lengths of all the colored triangles. So he can compute the sum of the squares of the sides of the colored triangles and hopefully the parameter a will disappear:

$$(\mathrm{AE}+\mathrm{EB})^2+\mathrm{BD}^2+\mathrm{BC}^2=(\sqrt{1-a^2}+\sqrt{3}a)^2+(1-4a^2)+(\sqrt{1-a^2}-\sqrt{3}a)^2=3$$

and it does, yielding a constant (that agrees with equation (5), since $r^2 = 1$). I have to admit this parameterization with a gives a slicker solution than mine with x.

[3] https://medium.com/u/bd708b43bfce