Two Men Meet

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This is another problem from the c.100AD Chinese mathematical work, *Jiŭ zhāng suàn shù* (The Nine Chapters on the Mathematical Art) found at the MAA *Convergence* website Convergence ([1]).

A square walled city measures 10 *li* on each side. At the center of each side is a gate. Two persons start walking from the center of the city. One walks out the south gate, the other the east gate. The person walking south proceeds an unknown number of *pu* then turns northeast and continues past the corner of the city until they meet the eastward traveler. The ratio of the speeds for the southward and eastward travelers is 5:3. How many *pu* did each walk before they met? [1 *li*= 300 *pu*]

Solution

Figure 1 shows a parmeterization of the problem. The ratio of the speed v_S of southward traveler to the speed v_E of the eastward traveler is

$$v_S / v_E = 5/3.$$

Note that $y/5 = 5/x$. Let $a = 5 + x$ and $b = 5 + y$. And let T be the time the travelers take from start to meeting. Then

$$\frac{v_S T}{v_E T} = \frac{5}{3} = \frac{b+d}{a}$$

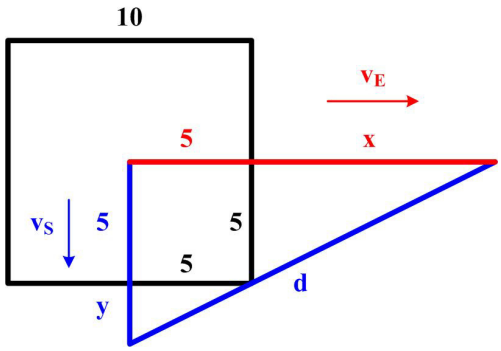

Figure 1

or $5a - 3b = 3d$

So $25a^2 - 30ab + 9b^2 = 9d^2 = 9(a^2 + b^2)$ (Pythagorean Theorem)

or $8a = 15b.$

So $8x - 15y = 35$

or $8x - 15(25/x) = 35$

So $8x^2 - 35x - 15\cdot 25 = 0$

Therefore,

$$x = \frac{35 \pm \sqrt{35^2 + 4\cdot 8\cdot 15\cdot 25}}{16} = \frac{35 \pm 5\sqrt{529}}{16} = \frac{35 \pm 5\cdot 23}{16} = \frac{75}{8}$$

So the distance the eastward traveler walked is

$$5 + x = (40 + 75)/8\ li = (115/8)\ 300\ pu = 4312\ ½\ pu$$

and the distance the southward traveler walked is

$$(5/3)\ 4312\ ½ = 7187\ ½\ pu.$$

Comment. Notice that it is believed that the Chinese knew about the Pythagorean Theorem result before the Greeks (as did the Babylonians c.1800BC ([3])) ([4]):

> The *Zhoubi Suanjing*, also known by many other names, is an ancient Chinese astronomical and mathematical work. The *Zhoubi* is most famous for its presentation of Chinese cosmology and a form of the Pythagorean theorem. It claims to present 246 problems worked out by the Duke of Zhou as well as members of his court, placing its composition during the 11th century BC. However, the present form of the book does not seem to be earlier than the Eastern Han (25–220 AD), with some additions and commentaries continuing to be added for several more centuries.[1] …

At one point during its discussion of the shadows cast by gnomons, the work presents a form of the Pythagorean theorem known as the gougu theorem (勾股定理) from the Chinese names—lit. 'hook' and 'thigh'—of the two sides of the carpenter or try square. In the 3rd century [AD], Zhao Shuang's commentary on the Zhoubi included a diagram effectively proving the theorem for the case of a 3-4-5 triangle [Figure 2], whence it can be generalized to all right triangles. The original text being ambiguous on its own, there is disagreement as to whether this proof was established by Zhao or merely represented an illustration of a previously understood concept earlier than Pythagoras [6th century BC]. Shang Gao concludes the gougu problem saying "He who understands the earth is a wise man, and he who understands the heavens is a sage. Knowledge is derived from the shadow [straight line], and the shadow is derived from the gnomon [right angle]. The combination of the gnomon with numbers is what guides and rules the ten thousand things."

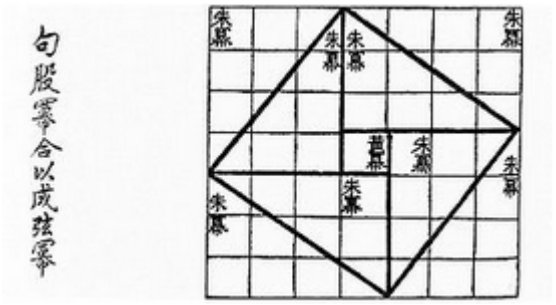

Figure 2

Presh Talwalkar had referred to this historical fact in the solution of one of his problems and so referred to the result as the Gougu Theorem in order to make the Chinese priority explicit.

References

[1] "Two Men Meet," *Convergence*, Mathematical Association of America, July 2006. From *Jiǔ zhāng suàn shù* (The Nine Chapters on the Mathematical Art) c. 100 AD ([2] p.1) (New link: https://old.maa.org/press/periodicals/convergence/two-men-meet).

[2] Cullen, Christopher, *The* Suàn shù shū 筭數書, *'Writings on reckoning': A translation of a Chinese mathematical collection of the second century BC, with explanatory commentary*, Needham Research Institute Working Papers: 1, Needham Research Institute, Cambridge, UK, 2004. (https://www.nri.org.uk/suanshushu.html)

[3] "Plimpton 322," *Wikipedia,* (https://en.wikipedia.org/wiki/Plimpton_322, retrieved 6/25/2024)

[4] "*Zhoubi Suanjing*," *Wikipedia,* (https://en.wikipedia.org/wiki/Zhoubi_Suanjing, retrieved 6/25/2024)

[1] Cullen ([2]) says the *Zhoubi* is 1st century BC, not 11th BC.