Sheep in Garden Problem

4 May 2025

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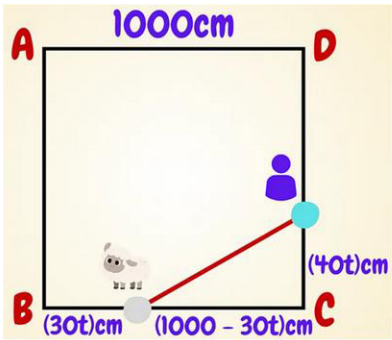

This is a slightly challenging problem from BL's Math Games.[1]

In a square garden ABCD of side 10m, a sheep sets off from B and moves along BC at 30cm per minute. At the same time, you set off from C and move along edge CD at 40cm per minute. The question is, what's the shortest distance between you and the sheep in meters?

This is somewhat an optimization problem because as you and the sheep move along the sides of the square at different rates, the distance in between varies as you can imagine.

There's at least one non-calculus solution and of course one calculus solution.

Solution (Vectors)

Let L be the distance between the sheep and me, v_s = 30 cm/min the constant speed of the sheep along BC, and v_m = 40 cm/min my constant speed along CD. Let $\mathbf{v_s}$ be the velocity vector of the sheep, and $\mathbf{v_m}$ the velocity vector of me. Let $\mathbf{v'_s}$ be the projection of the sheep's velocity vector along the line between the sheep and me, and $\mathbf{v'_m}$ be the projection of my velocity vector along that line (Figure 1 - Figure 3). Notice that the sheep's $\mathbf{v'_s}$ shrinks the distance and my $\mathbf{v'_m}$ expands the distance.

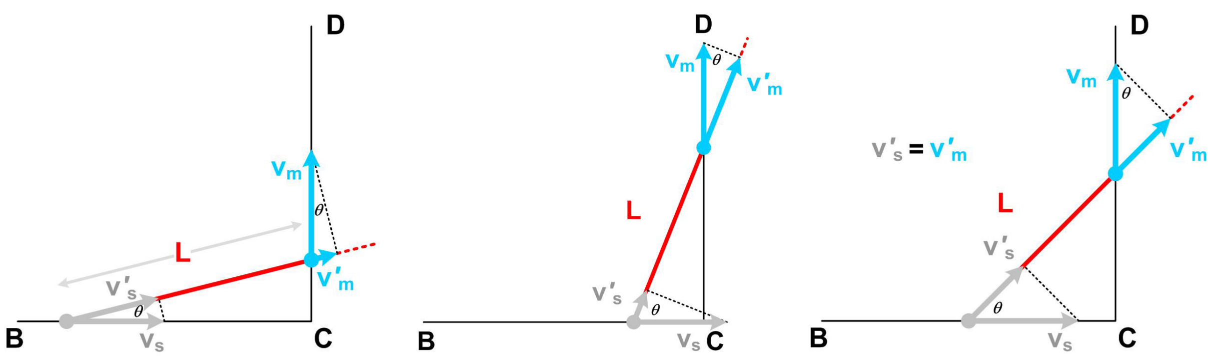

Figure 1 Shrinking Distance **Figure 2 Expanding Distance** **Figure 3 Minimal Distance**

In the beginning, $\mathbf{v'_s} > \mathbf{v'_m}$, so the total distance is shrinking (Figure 1), and toward the end, $\mathbf{v'_s} < \mathbf{v'_m}$, so the total distance is expanding (Figure 2). Therefore, at the moment when the total distance is neither shrinking nor expanding ($\mathbf{v'_s} = \mathbf{v'_m}$), that will be when the distance is minimal (Figure 3).

Now $v'_s = 30 \cos \theta$ and $v'_m = 40 \sin \theta$. Therefore, $v'_s = v'_m$ when $\tan \theta = 30/40 = ¾$. And so at that time t, the distances $v'_m t = 40t$ and $1000 - v'_s t = 1000 - 30t$ will also be in the ratio 3:4, that is,

$$\frac{40t}{1000 - 30t} = \frac{3}{4}$$

[1] 2 May 2025 (https://medium.com/math-games/in-a-square-garden-abcd-of-side-10m-a-sheep-sets-off-from-b-and-moves-along-bc-at-30cm-per-minute-54006628adc4)

Therefore, $t = 300/25 = 12$ min. And so $40t = 480$ cm. Since $\tan\theta = 3/4$ defines a 3-4-5 triangle, the minimal distance is

$$480 \text{ cm} \times 5/3 = 800 \text{ cm}$$

Solution (Calculus)

From the original description we have the distance between the sheep and me at any time t is

$$L(t) = \sqrt{(1000 - 30t)^2 + (40t)^2}$$

Taking derivatives,

$$L'(t) = \tfrac{1}{2}\frac{2(1000-30t)(-30) + 2(40t)(40)}{\sqrt{(1000-30t)^2 + (40t)^2}} = \frac{-30000 + 50^2 t}{\sqrt{(1000-30t)^2 + (40t)^2}}$$

So $L'(t) = 0 \Rightarrow t = 30000/2500 = 300/25 = 12$ min. Therefore, the minimal distance L is

$$L(12) = \sqrt{640^2 + 480^2} = \sqrt{(4\cdot 160)^2 + (3\cdot 160)^2} = \sqrt{(5\cdot 160)^2}\ \boxed{= 800 \text{ cm}}$$