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Calculus for Teachers: Whence Complex Numbers

David Bressoud, 15 July 2025

Last month I tried to convey why complex numbers should be of interest to someone teaching regular real-valued calculus.[1] This month I want to tackle an even more basic question: How did anybody even think to include the square roots of negative numbers as legitimate numbers? There is a wonderful chapter in Israel Kleiner's *Excursions in the History of Mathematics* titled "History of Complex Numbers with a Moral for Teachers" that I will be drawing on heavily for this month's column. He begins with an apt quote from the 12th-century Indian mathematician Bhaskara,

> The square of a positive number, also that of a negative number, is positive; and the square root of a positive number is twofold, positive and negative; there is no square root of a negative number, for a negative number is not a square.

No one wanted to think about square roots of negative numbers, but they were forced by Cardano's publication of a result stolen from Tartaglia and known secretly by del Ferro, a general method for finding exact values of the roots of a cubic equation. In modern notation, a solution of the equation is given by

$$x = \sqrt[3]{\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - \left(\frac{a}{2}\right)^3}} + \sqrt[3]{\frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - \left(\frac{a}{2}\right)^3}}.$$

Note that once you have one root of a cubic equation, say α, you can divide the polynomial by $x - \alpha$, reducing the problem to finding the roots of a quadratic polynomial.

In 1572 Rafael Bombelli published his textbook on algebra and tackled the problem of finding the solutions to $x^3 = 15x + 4$. Cardano's template yields

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.$$

But it is easy to see that $x = 4$ is a solution and therefore

$$x^3 - 15x - 4 = (x-4)(x^2 + 4x + 1).$$

The roots are 4 and $-2 \pm \sqrt{3}$. It seems that the complicated expression with square roots of –121 was supposed to equal 4, but how to make sense of it? Bombelli took audacious steps. He first assumed the $\sqrt{-121}$ could be written as $\sqrt{121} \cdot \sqrt{-1} = 11\sqrt{-1}$. Then he assumed that the first cube root could be written in the form $a + b\sqrt{-1}$. If we take the third power of $a + b\sqrt{-1}$ and assume that $\sqrt{-1} \cdot \sqrt{-1} = -1$, then

$$\left(a + b\sqrt{-1}\right)^3 = a^3 - 3ab^2 + (3a^2b - b^3)\sqrt{-1} = 2 + 11\sqrt{-1}.$$

Now we can solve for a and b: $a = 2$ and $b = 1$. The first cube root should equal $2 + \sqrt{-1}$ while the second would be $2 - \sqrt{-1}$. Their sum is 4. This all makes sense if one is allowed to work with $\sqrt{-1}$ as if it is a number.

Bombelli was far ahead of his time. European mathematicians were suspicious of negative numbers right through the 1600s. Great mathematicians including François Viète who worked in the

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late 1500s refused to use them. Even Newton was reluctant to accept a negative number as a solution. Square roots of negative numbers really were beyond the pale. It was Descartes in the early 1600s who, recognizing that he needed square roots of negative numbers when seeking the roots of cubic polynomials, expressed his distain for them when he coined the term *imaginary numbers*.

Kleiner describes a 1673 correspondence between Huygens and Leibniz. Leibniz had written to Huygens in a letter that included the identity

$$\sqrt{1+\sqrt{-3}}+\sqrt{1-\sqrt{-3}}=\sqrt{6}.$$

Huygens wrote back,

> The remark which you make concerning . . . imaginary quantities which, however, when added together yield a real quantity, is surprising and entirely novel. One would never have believed that √{1+ √{-3}} + √{1-√{-3}} make √6 and there is something hidden therein which is incomprehensible to me.

Leibniz and his student Johan Bernoulli would go on to employ these imaginary numbers to solve problems in integration. As we saw in last month's column,[2] Euler embraced them wholeheartedly, inventing the notation i for the square root of –1 and showing that

$$e^{i\theta}=\cos\theta+i\sin\theta.$$

This result, known as Euler's Formula, appears in his *Introduction to the Analysis of the Infinite*, a comprehensive treatment of calculus, including the derivation of infinite series. Euler proves his formula by observing that

$$\begin{aligned} e^{ix} &= 1+ix+\frac{i^2}{2!}x^2+\frac{i^3}{3!}x^3+\frac{i^4}{4!}x^4+\frac{i^5}{5!}x^5+\cdots \\ &= 1+ix-\frac{1}{2!}x^2-\frac{i}{3!}x^3+\frac{1}{4!}x^4+\frac{i}{5!}x^5+\cdots \\ &= 1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots \\ &\qquad +i\left(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\cdots\right) \\ &= \cos x+i\sin x. \end{aligned}$$

He gives a hint for how he may have discovered this. Earlier in his text he observes that imaginary numbers can be used to factor 1:

$$1=\cos^2 x+\sin^2 x=(\cos x+i\sin x)(\cos x-i\sin x).$$

This leads him to explore cos x + i sin x where he discovers that

$$(\cos x+i\sin x)(\cos y+i\sin y)=\cos(x+y)+i\sin(x+y).$$

Knowing that multiplication of two of these functions adds their arguments immediately suggests that this is an exponential function: $e^{\alpha x}\cdot e^{\alpha y}=e^{\alpha(x+y)}$. The only question is what is the value of α? Although he does not do this, he could have observed that the derivative of $e^{\alpha x}$ is $\alpha e^{\alpha x}$ and then seen that

$$\frac{d}{dx}(\cos x+i\sin x)=-\sin x+i\cos x=i(\cos x+i\sin x).$$

Therefore $\alpha= i$. This is not a proof, but it is very suggestive.

[2] https://maa.org/math-values/calculus-for-teachers-why-complex-numbers/

Throughout the 18th century as complex numbers became more acceptable, there was a growing awareness that every polynomial of degree $n > 0$ has exactly n roots (values at which it equals zero), some or all of which may be complex. In 1797, Carl Friedrich Gauss burst onto the mathematical scene with his doctoral thesis that provided the first essentially correct proof of this result. Note that it is enough to show that every nonconstant polynomial with real or complex coefficients has at least one complex root, call it α. We can divide our polynomial by $x - \alpha$ to get a polynomial of degree $n - 1$ that also must have a complex root, and so on. This has come to be known as the *Fundamental Theorem of Algebra*. One of its nicest proofs was given by my colleague Mike Hirschhorn in *The College Mathematics Journal*. His two-page proof is simple and very geometric. I strongly recommend my readers check it out: https://doi.org/10.1080/07468342.1998.11973954.

References

Hirschhorn, M. D. (1998). The Fundamental Theorem of Algebra. *The College Mathematics Journal*, **29**(4), 276–277. https://doi.org/10.1080/07468342.1998.11973954. [JOS: Behind a paywall]