100 Light Bulbs Puzzle

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Jim Stevenson

This is a classic puzzle[1] from Presh Talwalkar.

This puzzle has been asked as an interview question at tech companies like Google.

There are 100 lights numbered 1 to 100, all starting in the off position. There are also 100 people numbered 1 to 100. First, person 1 toggles every light switch (toggle means to change from off to on, or change from on to off). Then person 2 toggles every 2nd light switch, and so on, where person *i* toggles every i^{th} light switch. The last person is person 100 who toggles every 100th switch.

After all 100 people have passed, which light bulbs will be turned on?

I vaguely remembered the answer, which I confirmed after a few examples. But I didn't remember an exact proof, so I thought I would give it a try.

My Solution

It is clear that whether a light numbered *n* is off or on depends on how many smaller numbers divide it, that is, how many factors it has. Consider some examples.

number	factors	toggle state	number of factors	remark
1	1	on	odd	"square"
2	1, 2	off	even	prime
3	1, 3	off	even	prime
4	1, 2, 4	on	odd	square
5	1, 5	off	even	prime
6	1, 2, 3, 6	off	even	composite
7	1, 7	off	even	prime
8	1, 2, 4, 8	off	even	power of 2
9	1, 3, 9	on	odd	square

We see from the table that a light will only be on if its number has an odd number of factors. It looks like that will only occur with squared numbers, but we need to prove that.

Consider the unique factorization into powers of primes of the integer *n*:

$$n = 1 \cdot p_1^{n_1} p_2^{n_2} p_3^{n_3} \ldots p_k^{n_k}$$

For $i = 1, 2, \ldots, n_1$, p_1^i divides *n*, that is, it is a factor of *n*. So there are at least n_1 factors of *n*. Similarly for all the other prime factors. But so are their various products, which covers the cases where a prime factor is not included, that is, where the exponent on the prime is 0 (the prime factor

[1] 30 May 2025 (https://mindyourdecisions.com/blog/2025/05/30/100-light-bulbs-puzzle-2/)

can be thought of as 1). That means there are $(1 + n_1)\ (1 + n_2)\ (1 + n_3) \ldots (1 + n_k)$ possible factors, which includes 1 and n.

So the only way we can have an *odd* number of factors in a number is if all the primes have an *even* number of powers. That means n has to be a square:

$$n = 1 \cdot p_1^{n_1} p_2^{n_2} p_3^{n_3} \ldots p_k^{n_k} = 1 \cdot p_1^{2m_1} p_2^{2m_2} p_3^{2m_3} \ldots p_k^{2m_k} = 1 \cdot (p_1^{m_1} p_2^{m_2} p_3^{m_3} \ldots p_k^{m_k})^2$$

Talwalkar Solution

Talwalkar has a much simpler solution, which I had forgotten.

The lights turned on at the end are the square numbers 1, 4, 9, 16, 25, 36, 49, 64, 81, 100.

Person n will toggle all multiples of n. This equivalently means a light switch is toggled for each of its factors d.

Since each light starts in the off position, a light will be on if and only if it is toggled an odd number of times. Which numbers have an odd number of factors? We can work out some examples:

1 – factor 1
2 – factors 1, 2
3 – factors 1, 3
4 – factors 1, 2, 4
5 – factors 1, 5
6 – factors 1, 2, 3, 6
7 – factors 1, 7
8 – factors 1, 2, 4, 8
9 – factors 1, 3, 9

The numbers with an odd number of factors are 1, 4, 9. These are square numbers. And in fact this is a property of square numbers.

In a non-square number, every factor d pairs with a distinct other factor n/d, as $d(n/d) = n$.

But in a square number, the factor $\sqrt{n}$ pairs with itself, while the other factors do have distinct pairs, so the total number of factors is odd.

The square numbers from 1 to 100 are precisely the lights that will be on at the end. The lights turned on at the end area the square numbers 1, 4, 9, 16, 25, 36, 49, 64, 81, 100.

References

Geeks for geeks (100 doors puzzle) (https://www.geeksforgeeks.org/puzzle-16-100-doors/)

Joe Howard video (https://www.youtube.com/watch?v=SFNMflql5q8)