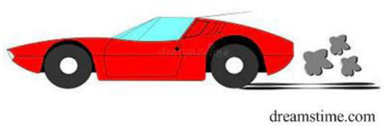
Unlawful Distance

10 January 2025

Jim Stevenson



This is a puzzle[1] from the A+Click site.

There is a fault with the cruise control on Hank's car such that the speed continuously and linearly increases with time. When he starts off the speed is set to exactly 60 mph. He is driving on a long straight route with the radio on at full blast and he is not paying any attention to his speed. After 3 hours he notices that his speed has now reached 80 mph. For how many miles did he drive above the state speed limit of 70 mph?

Answer Choices: 125 miles 112.5 miles 105 miles 99.5 miles

My Solution

One could proceed via calculus, but the linear aspects of the problem allow a more geometric approach. However, the student needs to realize the geometric meaning of distance = speed × time is an area.

So plotting the speed vs. time we see that the distance is given by the area under the curve (Figure 1). Since the speed curve is linear and since 70 mph is halfway between 60 and 80, the time the vehicle reaches that speed is halfway between 0 and 3 hours, namely 3/2 hours. Specifically, the speed is given by

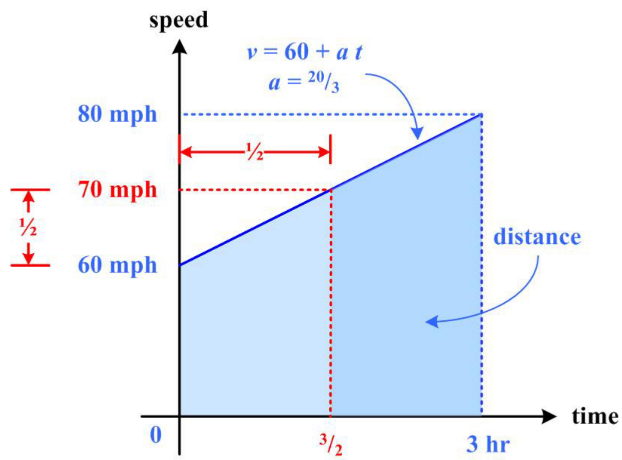


Figure 1

$$v = 60 + a\,t = 60 + (20/3)t$$

where a is the constant acceleration of 20/3 mph/hour. So when speed $v = 70$ mph, time $t = 3\cdot10/20 = 3/2$ hours.

The distance traveled after 70 mph is the darker blue area in the figure, which is the area of a trapezoid. So

$$\text{distance} = (3 - 3/2)(70 + 80)/2 = 75\cdot3/2 = 112.5 \text{ miles}$$

A+Click Solution

That was a tough one! The average speed is (60 + 80)/2 = 70 mph. We calculate the speed was achieved after 3/2 = 1.5 hours. He travels at an average speed of (70 + 80)/2 = 75 mph for 1.5 hours. The distance is speed × time, so

$$7.5 \times 1.5 = 75 + 37.5 = 112.5 \text{ miles}$$

I have a problem with the wording of this solution, especially the term "average". In the first instance the term is a red herring. It just so happens that the threshold speed of 70 mph is the average of 60 and 80, partly because the acceleration is constant (speed increases linearly). If the acceleration

[1] https://aplusclick.org/t.htm?level=12;q=2156

were not constant, then 70 would not be the average, nor would 1.5 hours be the time when the car reached 70, as I will show by an example below.

A second average is employed in working out the distance traveled after reaching 70 mph, namely (70 + 80)/2 = 75, which is then just multiplied by the remaining time to 3 hours, 1.5 hours. This second use is legitimate but employs circular reasoning in the case of non-constant acceleration, as I will show below.

Constant Acceleration.

Nevertheless, even in the linear case with constant acceleration the speed used may not be the average speed, and so it is misleading to use the phrase "average speed" as if the averaging is significant. For suppose Hank started at 50 mph instead of 60 mph. Then the equation for the speed would be

$$v = 50 + a\,t = 50 + 10t$$

Then 70 mph is not the average speed; 65 mph is. So we need to use the equation to find the time t when Hank reaches 70 mph (as we did in the first case as well). Therefore

$$70 = 50 + 10t \quad \Rightarrow \quad t = 2 \text{ hours}$$

The (trapezoidal) area under the linear curve from 2 hours to 3 hours is found in the same way as before, namely, (70 + 80)/2 mph × 1 hour = 75 miles. The reason is shown in Figure 2. Finding an average speed to multiply by the time traveled is geometrically equivalent to finding a rectangular area that equals the area under the curve. We can rotate the blue triangular area under the curve to coincide with the orange triangular area of the rectangle if the two triangles are congruent. For their vertical sides to be equal means the horizontal line marking the top of the rectangle must divide the vertical distance in half, and that turns out to be the average of the two heights of the trapezoidal area. In other words, we have an alternative derivation for the area of a trapezoid. This works no matter what the inclination of the linear equation is.

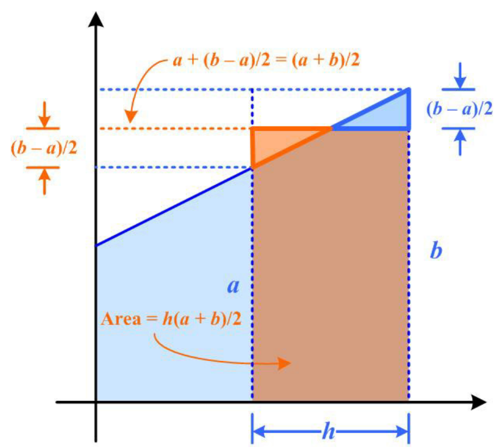


Figure 2

Non-Constant Acceleration.

If the acceleration were not constant, say $a = (20/9)t$, then the speed would be given by

$$v = 60 + a\,t = 60 + (20/9)t^2$$

Then the average speed would not be 70 mph but rather 66.7 mph (obtained by integration), and the time when the car reached 70 mph would not be 3/2 hours but rather $\sqrt{3} \approx 1.732$ hours (Figure 3). So again the "average" speed would not be relevant here, and it is misleading to use that term in the solution.

Now in evaluating the area under the speed curve from the time of 70 mph ($= 3/\sqrt{2} \approx 2.121$) to 3 hours one can multiply the average speed by the time

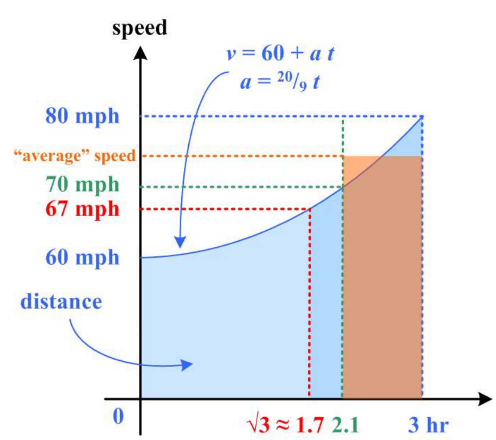


Figure 3

difference $(3 - 3/\sqrt{2})$ (Figure 3). But this does not avoid integration since the average speed is defined as the distance (i.e. integral of the speed) / $(3 - 3/\sqrt{2})$, and so is superfluous (or circular reasoning), since we would already have the distance by integration.