Rotating Square Problem

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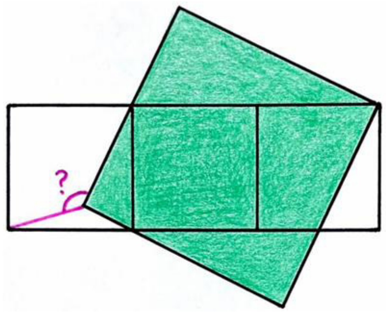

This is another Catriona Agg puzzle[1] that I again found somewhat challenging.

Four squares. What's the angle?

Visio showed me the answer fairly soon, but it took a bit to figure out a proof.

Solution

First, construct the perpendicular bisector of the top edge of the green square (Figure 1). The small orange and large green right triangles are similar. This means the hypotenuse of the orange triangle is half the length of the green triangle's hypotenuse, and so the vertex of the orange triangle coincides with the upper left corner of the small square. The two orange right triangles are also similar, having a second angle α in common. But they also have a common hypotenuse (from the small squares) and so are congruent. This means the long leg of the left orange triangle is half the length of the large square's edge. Therefore the short leg of the green right triangle is half the length of the long leg, and so tan α = ½.

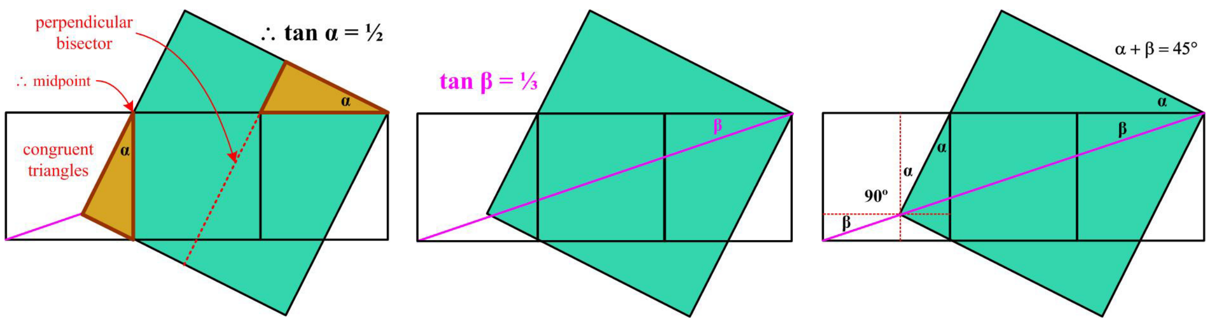

Figure 1 **Figure 2** **Figure 3**

Draw a (pink) line from the upper right corner of the right-most small square to the lower left corner of the left-most small square (Figure 2), and call the angle this line makes with the top edges of the three squares β. Then tan β = ⅓. Now

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}=1$$

Therefore, α + β = 45º. This means the pink line coincides with the diagonal of the large green square and so passes through its lower left corner (Figure 3).

That means the unknown angle is 90º + α + β = 90º + 45º = 135º.

Comment 1. After I figured out a proof for myself, I checked on Catriona Agg's Bluesky to see what others had done. There did not seem to be any simple, quick solutions. Many seemed to realize that the top edge of the squares bisected the left edge of the large square, but I wasn't able to see easily how, or if, they *proved* it. That had been my sticking point after Visio showed it *looked* to be true.

[1] June 7, 2025 (https://bsky.app/profile/catrionaagg.bsky.social/post/3lqywcmno6s2a)

Comment 2. The previous comment prompts me to mention something that befalls me all too often in these geometry problems, and which I believe others succumb to as well. If the picture rendering the problem is accurate enough, it will portray relationships that are true. A triangle may look like a right triangle, an intersection point may look like a midpoint, and so on—and they actually are what they appear. So in working out a derivation I am constantly slipping into thinking something is so that I haven't actually proved using plane geometry, or in desperation proved via trigonometry or analytic geometry.

That happened to me in spades in the "Curious Sunbeam Problem"[2] where there were so many linked true properties. That is, I had chains of statements A ⇒ B ⇒ C … L ⇒ M, which were actually equivalences, A ⇔ B ⇔ C … L ⇔ M, but I couldn't find a proof for A, or for that matter for any other of the statements independently. I couldn't find a starting point, so I had to resort to an unsatisfactory out-of-the-box start.

In this problem, I needed to have tan $\alpha = ½$, and it was obvious that the three squares intersected the large square at its midpoint, but how to prove it. As so many geometry "paradoxes" show,[3] you can't trust a drawing; you have to use logically proven statements, not pictures. Pictures are a guide, a seductive one, that you have to interrogate at each step: Why is this true?

By the way "Two Squares in a Circle"[4] is another example and Talwalkar even remarked that a number of solvers had assumed an "obvious" pivotal fact without proof. And it was that "fact" that I needed Bottema's Theorem to establish.

Sorry if I have been haranguing with the obvious.

Comment 3. Sorry again, we old codgers do go on. This situation I have been addressing really is at the heart of mathematical endeavors, where we are constantly trying to prove assertions we believe to be true. Often we are lulled into thinking we have succeeded, only to have our balloon popped when someone asks at a particular step, why?

This happened to Andrew Wiles who is credited with effectively proving the 300 year old Fermat's Last Theorem (the statement that no three positive integers a, b, and c satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than 2).[5] He dedicated more than six years in secret to the effort. When he finally presented his results in 1993, a flaw was discovered, which he labored for a year in vain to rectify. And then the solution arrived when he was about to give up, and he published the corrected result with his former student Anthony Taylor in 1995.

So this is why I consider plane geometry to be the gateway to doing real mathematics, and it is great that it is accessible to everyone with minimal tutelage.

[2] https://josmfs.net/2023/04/29/curious-sunbeam-problem/
[3] "Classic Geometry Paradox" (https://josmfs.net/2024/06/01/classic-geometry-paradox/)
[4] https://josmfs.net/2025/05/17/two-squares-in-a-circle/
[5] https://en.wikipedia.org/wiki/Andrew_Wiles