Contra Concrete Algebra

7 June 2025

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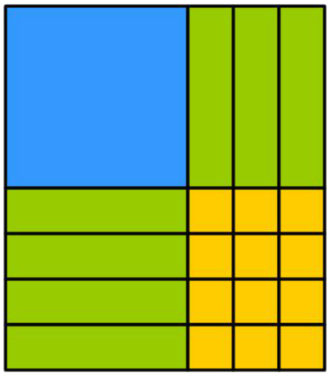

Apparently the pedagogical idea of wedding the understanding of algebra to concrete objects and actions that I thought might have been a fad some twenty years ago[1] is still employed today, as I discovered perusing Catriona Agg's Bluesky posts.

I want to be very clear that I am not denigrating the work and ideas of the dedicated teachers whose posts are collected on Agg's Bluesky account. I would have given anything to have such teachers for my high school math. I am especially impressed with the depth of material they are presenting that back in the day I only confronted in the early years of calculus classes in college. This requires very well-trained educators in the public (US) schools.

However, I would like to challenge this persistent pedagogical attachment to the concrete in teaching algebra, as I have done in my "Learning Mathematics" and "Cheshire Cat Paradigm"[2] posts.

This was the repost on Agg's Bluesky account that prompted my reaction ([1]):

In #MathsToday using my fave algebra tiles to factorise quadratics, so that students can visualise which factor pair they need.

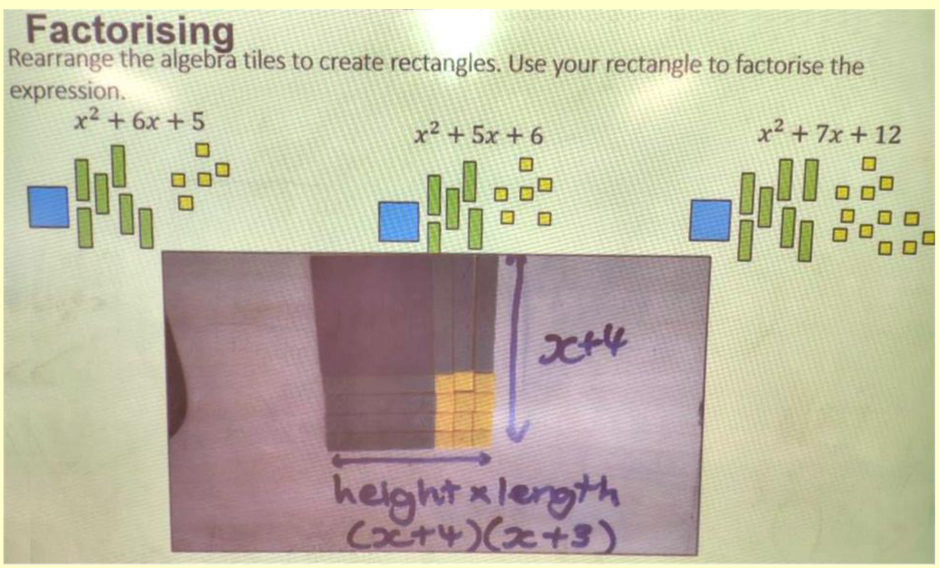

Figure 1 Algebra Tiles

So the idea seems to be for the student to factor a polynomial, they must *first* arrange the tiles in a rectangle in order to reflect the terms in the polynomial expression (Figure 2). It is a fun puzzle requiring some spatial acuity. But it is not the method of thought that will be required to handle modern algebra. It was the method used apparently over 4000 years ago before the advent of symbolic algebra in the Renaissance. In fact, math history books often show the geometric diagram they believe the Mesopotamians used to compute what we write in modern terms as $(a + b)^2 =$

[1] "Learning Mathematics" (https://josmfs.net/2024/05/04/learning-mathematics/)
[2] "Cheshire Cat Paradigm" (https://josmfs.net/2025/05/31/cheshire-cat-paradigm/)

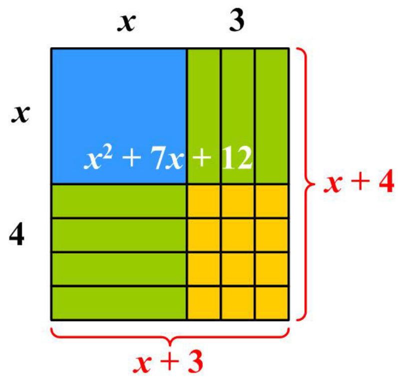

Figure 2 Algebra Tiles

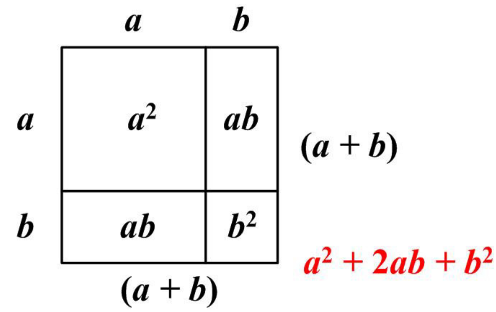

Figure 3 Babylonian "Computations"

$a^2 + 2ab + b^2$ (Figure 3). This is clearly the model for the algebra tiles approach to factoring polynomials.

But, as I tried to say in the Cheshire Cat post, math evolves. Before symbolic algebra ancients had only words, counting boards, and geometry to handle numbers and their properties. Once symbolic algebra arrived, doing that type of math became an exercise in manipulating symbols according to some rules.

Algebra became the art of "solving" equations, that is, finding the unknown number represented by a letter, such as x, that satisfied a polynomial equation, such as $x^2 + 7x + 12 = 0$. This was accomplished by factoring the polynomial, $x^2 + 7x + 12 = (x + 3)(x + 4) = 0$ and applying the property of numbers that if the product of two numbers equals zero, then one or both must be zero, that is, $ab = 0 \Rightarrow a = 0$ or $b = 0$ or both. So either $x = –3$ or $x = –4$. So factoring was intimately tied to finding roots.

Staying in the realm of numbers and roots, we see that for roots α, β,

$$(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta.$$

So in trying to factor an expression such as $x^2 + 7x + 12$, we see that +12 means the roots must have the same sign and be factors of 12, which are either 3, 4 or 2, 6 or 1, 12. Now we look at the coefficient of the x term and see it must be the negative sum of the roots. And since +7 is positive, the roots must be negative and sum to 7. And therefore we get the factorization $(x + 3)(x + 4)$.

Or we can forget the root origin of the coefficients in the polynomial $x^2 + Ax + B = (x + a)(x + b) = x^2 + (a + b)x + ab$ and just say that A must be the sum of the factors of B.

That is the mental process involved in factoring that stays in the realm of algebra (or arithmetic) and not geometry. Having to pause and go to a geometric model and solve a puzzle first is a distraction—and a throwback to an obsolete method. It does not offer any insight into the rule-based *algebraic* operation of factoring. It is kind of cool, after the fact, to go the other way from $x^2 + 7x + 12 = (x + 3)(x + 4)$ to a geometric picture using tiles, but it is not a viable math procedure now. I am a fan of the historical context of mathematical development and that it often follows the natural development of understanding in an individual, but it needs to be applied judiciously.

I also feel this concrete approach perpetuates counter-productively the notion that math, and especially algebra, is all based on concrete, visualizable things. That idea does not wear well when modern students are exposed to concepts in abstract algebra. One of the frustrating things, at least I found it so, is the difficulty in "picturing" abstract algebra. Vectors have a great visualizing aspect in their marriage to geometry and multivariable calculus (vector analysis), but as elements of abstract vector spaces involving the properties of linear transformations and matrices, such as Jordan decompositions, the imagery fades. There just aren't that many "pictures" inherent in abstract algebra—it's all objects and operations according to rules, like playing games. The proof in the Cheshire Cat post that (–1)(–1) = 1 is a great example. (That is why I personally find calculus and its intimate ties to geometry so much more understandable and satisfying—and why I am not an algebraist.)

References

[1] Lizi Pepper (https://bsky.app/profile/mathspeptalk.bsky.social/post/3lqptjod3w22u, retrieved 6/6/2025) as reposted by Catriona Agg (https://bsky.app/profile/catrionaagg.bsky.social) 3 June 2025.

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