Wittenbauer's Parallelogram

13 May 2025

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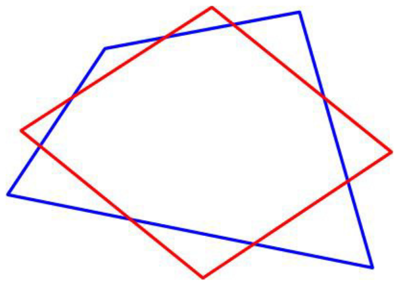

This is a lovely result from Futility Closet.[1]

Draw an arbitrary quadrilateral and divide each of its sides into three equal parts. Draw a line through adjacent points of trisection on either side of each vertex and you'll have a parallelogram.

Discovered by Austrian engineer Ferdinand Wittenbauer.

Find a proof.

Solution

Those with a good memory might recall that this statement is very similar to my post "Magic Parallelogram".[2] A look at the proof shows it could be tailored to prove this claim.

As before, add one of the diagonals of the original (blue) quadrilateral to form a triangle and represent the sides with vectors, as shown in Figure 1. Then we have again

$$t\mathbf{v} - t\mathbf{u} = t(\mathbf{v} - \mathbf{u}) \;\Rightarrow\; t\mathbf{v} - t\mathbf{u} \parallel \mathbf{v} - \mathbf{u},$$

only in this case $t = ⅓$ instead of $½$. This means the entire side of the red quadrilateral is parallel to the diagonal. Making the same argument with the opposite vertex of the blue quadrilateral means the two sides of the red quadrilateral are parallel to each other.

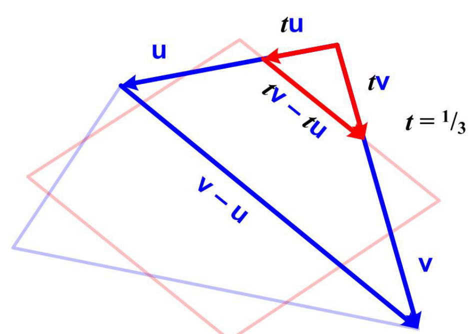

Figure 1

Similarly, adding the other diagonal to the blue quadrilateral shows that the other two sides of the red quadrilateral are parallel.

Therefore the red quadrilateral must be a parallelogram.

Comment. I suppose I should prove this last statement. Suppose the opposite sides of a quadrilateral are parallel (Figure 2). Draw one of the diagonals. Then the corresponding angles are all equal, so the two triangles formed by the diagonal are similar. But they have one side in common, the diagonal, so they must be congruent. Therefore their corresponding sides are equal.

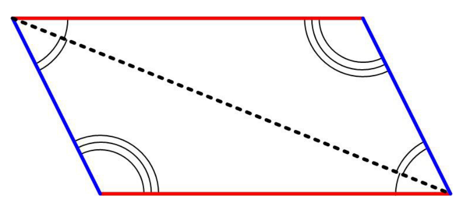

Figure 2

[1] 12 May 2025 (https://www.futilitycloset.com/2025/05/12/wittenbauers-parallelogram/)
[2] http://josmfs.net/2019/04/05/magic-parallelogram/