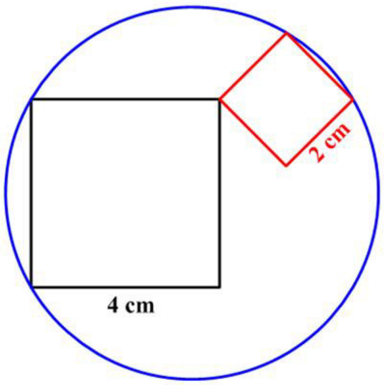
Two Squares in a Circle

14 May 2025

Jim Stevenson



This puzzle, from another set of seven challenges[1] assembled by Presh Talwalkar, turned out to be very challenging for me.

This is a fun problem I saw on Reddit AskMath. A circle contains two squares with sides of 4 and 2 cm that overlap at one point, as shown. What is the area of the circle?

This took me quite a while to figure out, but I relied on another problem I had posted earlier.

My Solution

In trying to construct the diagram I discovered there seemed to be only one arrangement of the squares so that their vertices lay on a circle. Furthermore it looked like the top edge of the black square extended would be collinear with the diagonal of the red square. And similarly the left edge of the red square extended seemed to be collinear with the diagonal of the black square. But how to prove it.

Then the canted configuration of the squares reminded me of "Bottema's Theorem".[2] So I drew two (pink) lines joining the vertices of the squares as shown in Figure 1. It turns out, from the *proof* of Bottema's Theorem, that the (fixed) midpoint of the line joining the extreme vertices lies on the perpendicular bisector of the (pink dashed) baseline of the triangle.

The center of the circle lies on this perpendicular bisector. But it also lies on the perpendicular bisector of the solid pink line, and therefore on the intersection of that bisector and the dashed

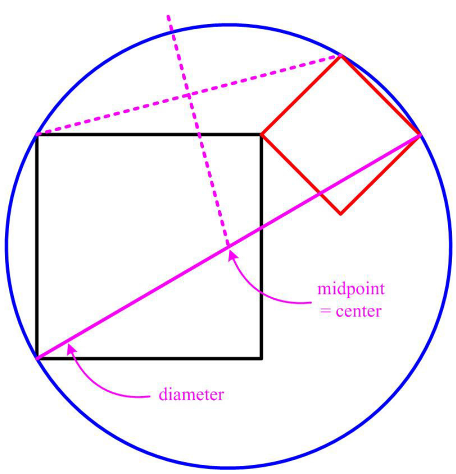


Figure 1 Bottema's Theorem (Proof)

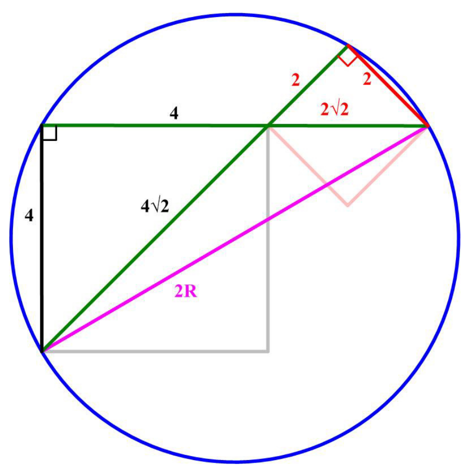


Figure 2

[1] 25 April 2025 (https://mindyourdecisions.com/blog/2025/04/25/7-puzzles-that-will-test-your-skills/)

[2] "Bottema's Theorem" (https://josmfs.net/2021/07/03/bottemas-theorem/)

perpendicular, which happens to be the midpoint of the solid pink line. This means the midpoint of the solid pink line is the center of the circle and so *the line is a diameter of the circle*.

Now join the upper left vertex of the black square with the furthest right vertex of the red square (Figure 2). This forms a triangle with the solid pink line, and so must be a right triangle, since the pink line is a diameter. That means the line is collinear with the top edge of the black square. Similarly, the line joining the top vertex of the red square with the lower left vertex of the black square must be collinear with the red square's edge.

From either of these right triangles we can compute the diameter of the circle via the Pythagorean Theorem:

$$4\,R^2 = (4 + 4\sqrt{2})^2 + 4^2 = 8(5 + 2\sqrt{2}).$$

So the area of the circle is

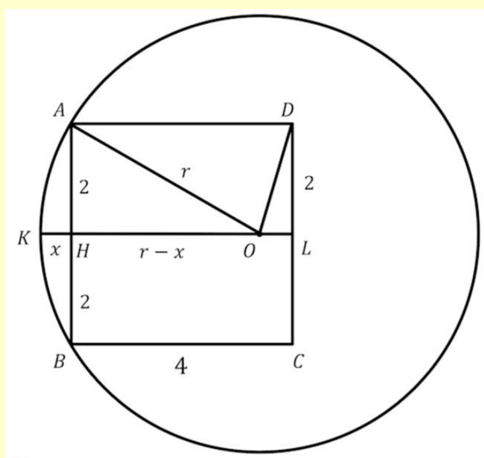
$$\pi R^2 = \pi\,(10 + 4\sqrt{2})$$

Comment. The heavy lifting in this solution, then, was done by Bottema's Theorem, in particular, the proof.

Talwalkar Solution

The credit for the solution goes to u/testtest26.[3]

Let the larger square be *ABCD* with *AB* a chord of the circle. Construct the perpendicular bisector of *AB* which passes through the circle's center, call that *O*. Let *K* be the endpoint of the perpendicular bisector on the circle, *H* be the intersection with *AB* and *L* be the endpoint on *CD*. Let the circle's radius be *r*, and let $KH = x$. The side *AB* is bisected by *KH* so $AH = 4/2 = 2$. Also construct *OD*.



We have 2 right triangles *OAH* and *ODL*. We thus get the two equations:

$$AH^2 + OH^2 = OA^2$$

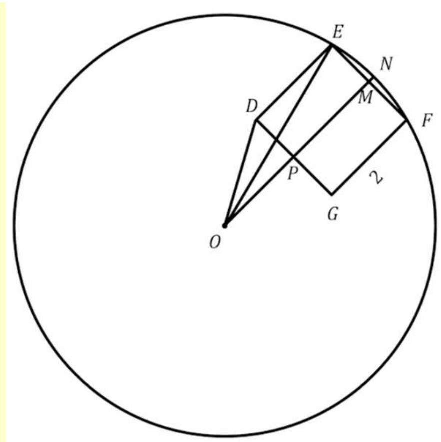
$$2^2 + (r - x)^2 = r^2$$

$$DL^2 + OL^2 = OD^2$$

$$2^2 + (4 + x - r)^2 = OD^2$$

We will do a similar construction with the other square.

[3] https://www.reddit.com/r/askmath/comments/1iwxdq2/comment/mehv44p/



Let $MN = y$, and we have $EM = 2/2 = 1$. Thus we get the two equations:

$$EM^2 + OM^2 = OE^2$$

$$1^2 + (r - y)^2 = r^2$$

$$DP^2 + OP^2 = OD^2$$

$$1^2 + (r - y - 2)^2 = OD^2$$

We have 4 equations. The video shows how to solve it by hand (it is quite tedious—you solve for $r - x$ and $r - y$ in the first equation of each set, then substitute to the second equations and keep squaring to remove all radicals).

For this post I will shortcut with WolframAlpha.[4]

Input

$\{2^2 + (r - x)^2 = r^2, 2^2 + (4 + x - r)^2 = d^2, 1^2 + (r - y)^2 = r^2, 1^2 + (r - y - 2)^2 = d^2\}$

Alternate form

$\{2rx = x^2 + 4, (-r + x + 4)^2 + 4 = d^2, 2ry = y^2 + 1, (-r + y + 2)^2 + 1 = d^2\}$

$\{rx = \frac{x^2}{2} + 2, -d^2 + r^2 + r(-2x - 8) + x^2 + 8x = -20,$
$ry = \frac{y^2}{2} + \frac{1}{2}, -d^2 + r^2 + r(-2y - 4) + y^2 + 4y = -5\}$

Expanded form — Step-by-step solution

$\{-2rx + x^2 + 4 = 0, -d^2 + r^2 - 2rx - 8r + x^2 + 8x + 20 = 0,$
$-2ry + y^2 + 1 = 0, -d^2 + r^2 - 2ry - 4r + y^2 + 4y + 5 = 0\}$

Solutions — More roots — More digits — Exact forms — Step-by-step solution

$d \approx -2.0840, \quad r \approx -3.9569, \quad x \approx -7.3711, \quad y \approx -7.7853$

$d \approx -2.0840, \quad r \approx 3.9569, \quad x \approx 0.54266, \quad y \approx 0.12845$

$d \approx 2.0840, \quad r \approx -3.9569, \quad x \approx -7.3711, \quad y \approx -7.7853$

$d \approx 2.0840, \quad r \approx 3.9569, \quad x \approx 0.54266, \quad y \approx 0.12845$

$d \approx -3.9569, \quad r \approx -2.0840, \quad x \approx -2.6698, \quad y \approx -0.25559$

[4] https://www.wolframalpha.com/input?i=2%5E2%2B%28r-x%29%5E2%3Dr%5E2%2C+2%5E2%2B%284+%2B+x-r%29%5E2%3Dd%5E2%2C+1%5E2%2B%28r-y%29%5E2%3Dr%5E2%2C+1%5E2%2B%28r-y-2%29%5E2%3Dd%5E2

Taking only positive distances there are 2 possible solutions for the radius:

$$r = \sqrt{2(5 - 2\sqrt{2})}$$

$$r \approx 2.084$$

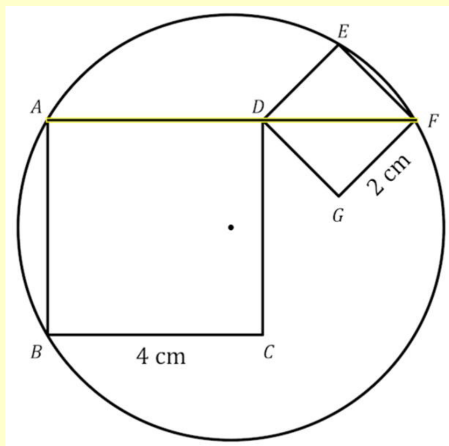
$$r = \sqrt{2(5 + 2\sqrt{2})}$$

$$r \approx 3.9569$$

For the circle to contain squares of sides 4 and 2, the circle must have an area larger than the area of the squares which is 16 + 4 = 20. This is only possible in the second solution case. Thus we have the area of the circle is:

$$\pi r^2 = \pi(10 + 4\sqrt{2}) \approx 49.19 \text{ cm}^2$$

I should mention many people think they have found an easier solution by assuming *A*, *D* and *F* are collinear.



I agree the problem will be relatively simple with this step. However, this step cannot be assumed, and I have yet to see an easy to prove it.[5] If you do find a solution, I suggest you post it to Reddit in any of the threads linked below so everyone can learn.

Problem:

https://www.reddit.com/r/askmath/comments/1iwxdq2/find_the_area_of_the_circle/

solution by u/testtest26:

https://www.reddit.com/r/askmath/comments/1iwxdq2/comment/mehv44p/

hard to prove straight line:

https://www.reddit.com/r/maths/comments/15ubhwk/geometry_question/

[5] JOS: This is what I did in my solution.