Cheshire Cat Paradigm

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I have been meaning to focus on this aspect of mathematics for some time. It is a topic I elaborated in my "Angular Momentum"[1] post. But I also think it has something to do with the difficulties that normal folks have with elementary math, in particular, numbers. I thought I would dub it the Cheshire Cat Paradigm, involving the Cheshire Cat's grin.

Just to recall the salient part of the conversation in Lewis Carroll's *Alice in Wonderland* between Alice and the Cheshire Cat ([1]):

> … As she said this, she looked up, and there was the Cat again, sitting on a branch of a tree.
>
> 'Did you say pig, or fig?' said the Cat.
>
> 'I said pig,' replied Alice; 'and I wish you wouldn't keep appearing and vanishing so suddenly: you make one quite giddy.'
>
> 'All right,' said the Cat; and this time it vanished quite slowly, beginning with the end of the tail, and ending with the grin, which remained some time after the rest of it had gone.
>
> 'Well! I've often seen a cat without a grin,' thought Alice; 'but a grin without a cat! It's the most curious thing I ever saw in my life!'

Angular Momentum

I will start with the aforementioned angular momentum. It began with rotating objects, such as spoked wheels, and the introduction of the notion of torque τ, a transverse rotational force, caused by pushing one of the spokes around the hub to start the wheel rotating. The spoke acted as a lever arm: the longer the spoke, the more the force required to get the wheel turning, and also the heavier the rim, the more force required. So torque depends on the mass of the rim and how far it is from the hub.

As we saw in the article, Feynman showed that torque represented the time rate of change of something analogous to a linear force being the time rate of change of linear momentum, mv, where m is the mass and v the velocity of the object. So Feynman dubbed this thing "angular momentum" L and torque the time rate of change of angular momentum, $\tau = dL/dt$. Therefore once the wheel is spinning and no further torque is exerted, $\tau = dL/dt = 0$, and so angular momentum L is constant, or conserved.

Then this mathematical formulation was applied to a setting where there were no mechanical connections to exert forces, namely, the motion of planets around the sun. Kepler knew there was some sort of magnetism that drew planets radially toward the sun and he posited some sort of transverse lever arm type of force, also exerted by the sun, that caused the planets' circular motion. Then Newton replaced the centripetal "magnetic" force with his "gravitational" force (whose cause he

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[1] "Angular Momentum" (http://josmfs.net/2019/01/09/angular-momentum/)

had no idea about except it had something to do with the size of the sun and planets) and replaced the transverse lever arm force with “inertia” (his first Law of Motion), which essentially said a planet is moving transversally because it is already moving transversally.

The equations for torque and angular momentum turned out to be the same, and so it must be the same phenomena, only this time there were no mechanical linkages to explain it—the Cheshire Cat’s grin without the cat!

Electromagnetic Waves

This disappearing act happened again in the last half of the 19th century. In the 1860s Maxwell linked the phenomena of electricity and magnetism in a set of partial differential equations that showed these phenomena satisfied a partial differential wave equation, where these “electromagnetic” waves would propagate at a speed c in a vacuum that turned out to be the speed of light.

Now wave phenomena had long been captured in mathematics by circular or trigonometric functions that represented the periodic or oscillatory behavior of waves. The same mathematics could describe the longitudinal waves caused by sound traveling through the air as vibrations or transversal waves caused by the up and down motion of water. Electromagnetic waves belonged to the category of transversal waves.

The question was, what was the medium that supported electromagnetic waves, like air for sound and water for ocean waves? It was dubbed the “aether”. The behavior of a wave is affected by the medium through which it is traversing. The bending of a pencil inserted into a bowl of water comes from the different speeds of light in air and water. The Doppler effect in a sound wave occurs when the source of sound is moving toward a receiver and then away. The sound wave is traveling at the same speed in air, but an approaching source will cause the sound pulses to come together more quickly and produce a higher frequency in the received sound. Then a receding source will cause the pulses to spread out and produce a lower frequency at the receiver.

So what is the effect on light traveling in different directions in the aether as the earth moves through space (supposedly filled with the aether) (Figure 1)? This was answered by the famous Michelson-Morley experiments culminating in 1887 ([2]).

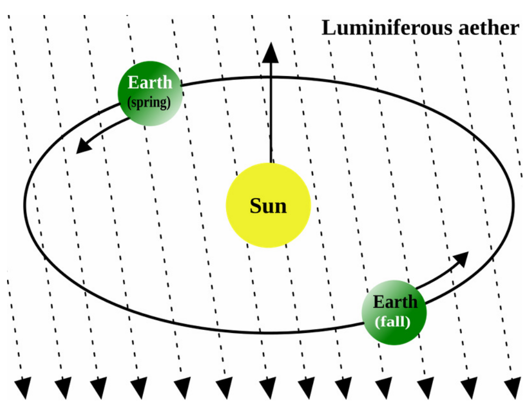

Figure 1

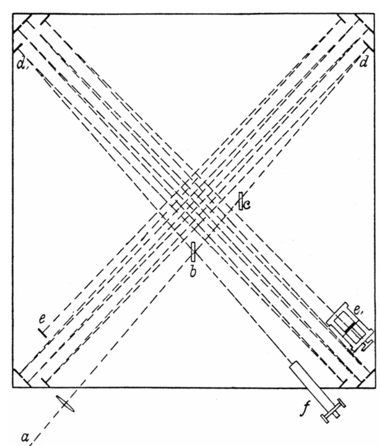

Figure 2 Michelson-Morley Experiment

Essentially the idea was to send a single beam of light from a source (a in Figure 2), through a set of beam splitters and mirrors that would send replicas of the beam in orthogonal directions, over the same path length to be recombined at a single receiver (f in Figure 2). The light beams should experience a Doppler effect along the different paths for the earth moving through the aether. This would be detected by a frequency mismatch or interference pattern of the combined beams at the receiver.

None was detected. It was as if the aether did not exist.

Slight digression: Since the frequency of an oscillation is proportional to the reciprocal of the period, the Doppler effect (change of frequency between source and receiver) is equivalently a change in the measures of time differences (periods)—time dilation. So Lorentz proposed an effective time dilation equation that would nullify the supposed Doppler effect. But he did not take the step of actually thinking that time might be measured differently at two frames of reference moving with respect to one another (no absolute time). Einstein did—and thus was born Special Relativity, which claimed that the speed of light would be measured the same in all frames of reference moving at a constant velocity with respect to one another.

In any case, the result was that for all practical purposes the aether did not exist. There was no medium whose vibrations generated electromagnetic waves—the Cheshire Cat's grin without the cat strikes again!

Numbers

So we come to a simpler, more accessible example. As we have seen, for millennia societies had ways of counting things and these resulting numbers were called "counting numbers". They had different ways of marking these numbers when writing was invented. When the Greeks were bringing numbers to geometric objects that were abstracted from the measurements of real objects, they encountered a difficulty with measuring the hypotenuse of an isosceles right triangle, due to the Pythagorean Theorem. There was no unit of length-measurement that was so small that both the equal legs of the triangle and the hypotenuse could be measured by counting-numbers of these units—the legs and hypotenuse were incommensurable. (In more modern parlance this is equivalent to saying the square root of 2 is not rational.) So the Greeks relegated these incommensurable numbers to measures on geometric figures. That is, whenever they encountered them, it was solely as constructions in geometric figures; they were not considered true numbers that measured multiplicity.

One could add and subtract counting numbers, and multiply them as well, but without the existence of expendable paper, these operations were generally carried out on mechanical devices, such as counting boards, and later abacuses.

This state of affairs persisted in the West up to the Middle Ages when the fruits of Indian and Chinese mathematics filtered through the contributions of the Islamic world were introduced by works such as the *Liber Abaci* by Leonardo of Pisa (Fibonacci). These new ideas brought with them new numbers: zero, negative numbers, and numerator-denominator fractions (common fractions)—along with the numerals for 10 digits and a base-10 number representation.

Zero was coerced into being a counting number by imagining it measured the amount of "nothing", such as when all objects in a bowl were subtracted leaving nothing. Notions of what is a number got a bit more wobbly with negative numbers. They seemed to count some things, such as the amount of debt one had, or the number of feet below a marker that water had dropped, but adding and multiplying them like counting numbers was strange—still, it worked! There was some confusion about what multiplying two negative numbers to produce a positive number meant, but it led to consistent results.

What do I mean by consistent results? The actions followed certain rules or properties:

Rule 1. It didn't matter what order you added or multiplied three numbers, the result was the same.

Associativity: $(m + n) + k = m + (n + k)$ and $(m \times n) \times k = m \times (n \times k)$

Rule 2. It didn't matter what order you added or multiplied two numbers, the result was the same.

Commutativity: $m + n = n + m$ and $m \times n = n \times m$

Rule 3. There were identity numbers that didn't change the result when added to or multiplied a number.

Additive Identity 0: $0 + n = n$ Multiplicative Identity 1: $1 \times n = n$

Rule 4. For every number there was an additive and multiplicative inverse (except for 0).

Additive Inverse $-n$: $n + (-n) = 0$ Multiplicative Inverse $1/n$, $n \neq 0$: $n \times (1/n) = 1$

Rule 5. Addition "gets along with" multiplication.

Distributivity: $m \times (n + k) = (m \times n) + (m \times k)$

These "rules" hid some things that are not easily visualized with counting numbers, if at all. If we imagine multiplication is defined as repeated addition, such as $4 \times 5 = 5 + 5 + 5 + 5$, then it is a bit surprising at first that $5 \times 4 = 4 + 4 + 4 + 4 + 4$ would give the same result (20). And since $0 + 0 = 0$, it makes sense that $4 \times 0 = 0 + 0 + 0 + 0 = 0$. But $0 \times 4 = 0$? That is a bit of a stretch. But if we keep the commutativity Rule 2, then this is what we must have. The same weirdness happens with multiplying positive and negative numbers. $4 \times (-3) = (-3) + (-3) + (-3) + (-3) = -12$, but $(-3) \times 4 = -12$? It does if we want to keep commutativity.

How about the dreaded multiplication of two negatives? If we want to keep the distributivity Rule 5 and all the other rules, then

$$1 = 1 + 0 = 1 + (-1) \times 0 = 1 + (-1) \times (1 + (-1)) = 1 + (-1 \times 1) + (-1) \times (-1) = 0 + (-1) \times (-1)$$

So $$(-1) \times (-1) = 1$$

Noting that $-m = (-1) \times m$ and using the commutativity rules, we must have the product of two negative numbers be positive:

$$(-m) \times (-n) = (-1) \times (-1) \times m \times n = 1 \times m \times n = m \times n$$

Why would we want to have these rules take priority over being able to visualize a mirrored concrete behavior? As we saw in the timeline of the evolution of mathematics,[2] symbolic algebra was developing about the same time as these new notions of numbers were infiltrating the West. This meant that equations like $x^2 + 3x = 4$ (there was still a reluctance to consider negative numbers real) were showing up where x represented an unknown number. They wanted to find the value of x, but realized it might not be a counting number. It might be one of these new numbers (negatives or rationals, or even irrationals, or even more disturbingly, square roots of negative numbers). So they wanted whatever the answer might be to follow the same rules above derived from the counting numbers—even if that meant the operations made no direct sense as counting numbers!

So a mathematical abstraction for what a number means (set of rules) replaced the original concrete reality of counting things—again the Cheshire Cat grin without the cat.

I believe this evolution of the idea of number that leaves the reality of counting behind is the main source of difficulty for human beings to learn mathematics. That is also why I am reluctant to have math education persist in trying to maintain this concrete attachment.[3] Rather it should inculcate this evolution—of course starting with the concrete, but then acknowledging that the idea of number evolved to an abstracted structure that has amazing and mysterious echoes in all sorts of physical phenomena that did not involve counting.

We are truly living in an imaginary world that seems to mirror whatever reality is. It is the adult version of the *Just So* stories ([3]), and should be learned accordingly.

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[2] "Timelines" (https://josmfs.net/symbolic-algebra-timelines/)
[3] "Learning Mathematics" (https://josmfs.net/2024/05/04/learning-mathematics/)

References

[1] Carroll, Lewis, *Alice's Adventures In Wonderland with Forty-Two Illustrations by John Tenniel*, Macmillan & Co., London, 1865, D. Appleton & Co., New York, 1866, 1898. pp.93-94. (https://archive.org/details/carroll-1898-alice-in-wonderland/Carroll_1898_Alice_in_Wonderland/page/2/mode/2up, retrieved 5/24/2025)

[2] "Michelson-Morley Experiment", *Wikipedia*, (https://en.wikipedia.org/wiki/Michelson%E2%80%93Morley_experiment, retrieved 5/24/2025)

[3] Kipling, Rudyard, *Just So Stories*, Pictures By Joseph M. Gleeson, Doubleday Page & Company, 1912. (https://www.gutenberg.org/files/32488/32488-h/32488-h.htm, retrieved 11/16/2024)