Mystery Curve

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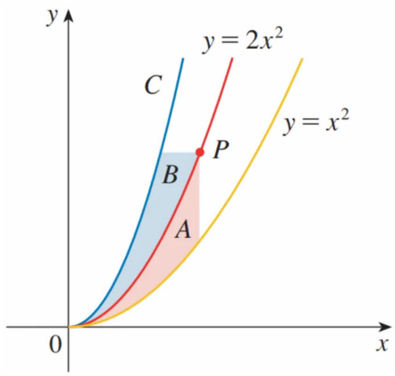
Jim Stevenson



Here is another problem from BL's Weekly Math Games.[1]

For every point P on $y = 2x^2$, areas A and B are equal. Find the equation for curve C.

Solution

Let $y = f(x)$ be the unknown function for the curve C. From the diagram we can assume f is invertible in the region of interest (first quadrant). Let $x = g(y)$ be the inverse function. That is, $y = f(g(y))$ and $x = g(f(x))$.

Then the areas (Figure 1) can be given by the integrals

$$A(a) = \int_0^a \left(2x^2 - x^2\right)dx = \int_0^a x^2 dx$$

and

$$B(a) = \int_0^{2a^2} \left(\sqrt{\frac{y}{2}} - g(y)\right)dy \,.$$

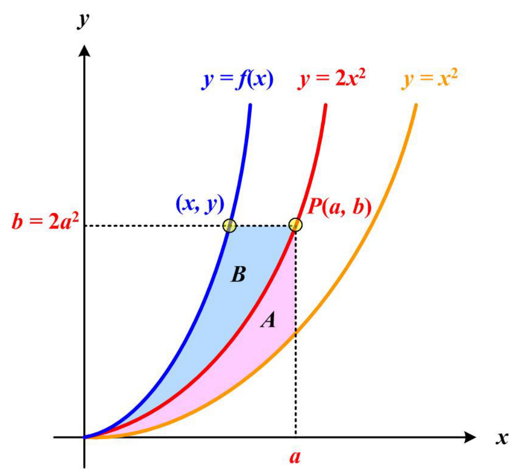


Figure 1

So $A(a) = B(a)$ means their derivatives with respect to a are equal as well: $A'(a) = B'(a)$. So

$$a^2 = \left[\sqrt{\frac{2a^2}{2}} - g(2a^2)\right]4a$$

or, since $a > 0$,

$$g(2a^2) = 3a/4.$$

And so

$$2a^2 = f(3a/4).$$

Let $x = 3a/4$. Then $a = 4x/3$ and

$$y = f(x) = 2(16x^2/9) = (32/9)\,x^2.$$

[1] 28 September 2024 (https://medium.com/bellas-weekly-math-games/can-you-find-the-equation-66a8fe7c7644)