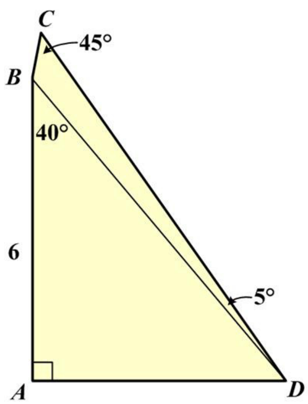
Chinese Quadrilateral Puzzle

29 January 2025

Jim Stevenson



This is another intimidating puzzle[1] from Presh Talwalkar:

Thanks to Eric from Miami for suggesting this problem and sending a solution!

From a 5th grade Chinese textbook: In the quadrilateral *ABCD*, angle *A* = 90°, angle *ABD* = 40°, angle *BDC* = 5°, angle *C* = 45°, and the length of *AB* is 6. Find the area of the quadrilateral *ABCD*.

My Solution

I'm a bit dubious that the average 5th grader can handle the sophistication of a logical geometric proof, which eludes even many adults. If true, the times have certainly changed.

My only consolation in this problem is that I seemed to have found another simple solution for the problem that doesn't involve trigonometry. (Although it is not quite as simple as Talwalkar's second solution perhaps, it still seems easier to think of, at least to me).

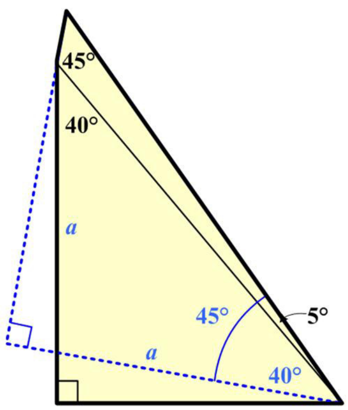


Figure 1

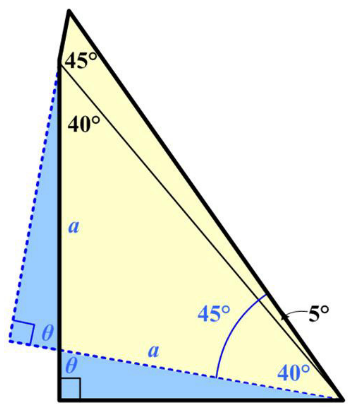


Figure 2

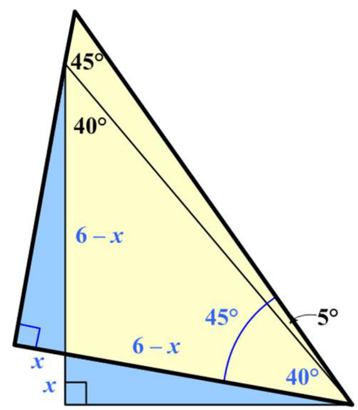


Figure 3

The first thing I wondered when I looked at the funny bend at vertex *B* was what would happen if you extended the line *CB* and then drew a second line perpendicular to this extended line to form a 45° isosceles right triangle (Figure 1)? Well, we get another isosceles triangle with base angles 40° and sides *a*. In addition we get two (blue) right triangles that have an angle θ in common and so are similar. But they also have equal hypotenuses *a* and so are congruent (Figure 3). This means the area of the 45° isosceles right triangle is the same as the area of the original quadrilateral.

If the short leg of the blue right triangles is of length x, then from the problem statement the hypotenuse *a* is of length $6 - x$. This implies the length of each leg of the 45° isosceles right triangle is $(6 - x) + x = 6$, and that means the area of this triangle (and the original quadrilateral) is

$$\tfrac{1}{2}\, 6 \times 6 = 18.$$

[1] Puzzle 3, 28 January 2025 (https://mindyourdecisions.com/blog/2025/01/28/3-problems-from-china-with-outside-the-box-thinking/#more-37545)

Talwakar Solutions

I would have solved the problem using trigonometry, which is beyond what a 5th grader would have learned. Still, I will present this complicated solution to show it is possible. Then I will reveal the neat trick which is how 5th graders were expected to solve it.

First we will calculate some angles and lengths (Figure 4).

angle ADB = 90° – 40° = 50°
angle DBC = 180° – 45° – 5° = 130°
AD = 6 tan 40°
BD = 6/cos 40°

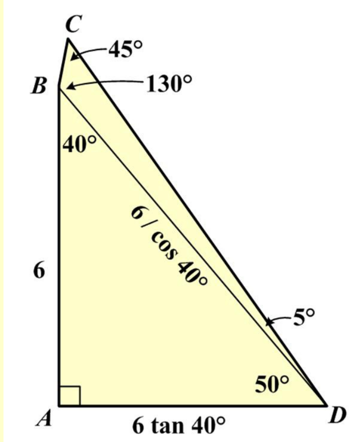


Figure 4

Then we can use the law of sines to calculate CB:

CB/BD = (sin 5°)/(sin 45°)
CB = (6/cos 40°)(sin 5°)/(sin 45°)

We can then calculate the areas of triangles ABD and CBD. Triangle ABD is a right triangle, so its area is half its base times its height.

area ABD = 6(6 tan 40°)/2

Triangle CBD can be solved using the formula (side 1)(side 2)(sine angle between)/2. So we get:

area CBD [2] = (6/cos 40°)[(sin 5°) / (sin 45°)] (6/cos 40°) (sin 130°) / 2

The total area is the sum of these, which can be found numerically on a calculator.[3] Remarkably, the answer simplifies tremendously—it is an integer!

area ABCD = 18

So now the question is, how would 5th graders have found the answer? Here's the incredible method.

The trick: flip triangle CBD

If we flip triangle CBD, side BD remains in the same position, but we flip the other sides. Here is the shape we end up creating (Figure 5):

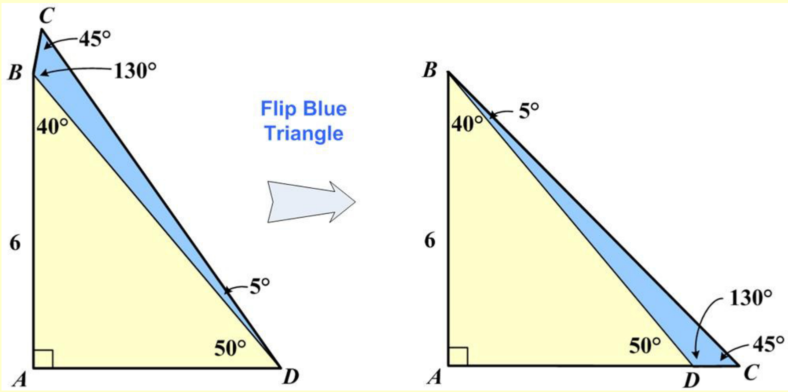


Figure 5

[2] JOS: Corrected. The second (6/cos 40°) was omitted from the original. It's easy to make mistakes when things get overly complicated.

[3] http://www.wolframalpha.com/input/?i=(6+tan+(40+deg))*6%2F2%2B(6%2Fcos+(40+deg))%5E2*(sin+(130+deg))*(sin+(5+deg))%2F(2*(sin+(45+deg))

Now we can calculate some angles in this new figure:

angle ADC = 50° + 130° = 180° (a straight line)
angle ABC = 40° + 5° = 45°
angle ACB = 45°

Notice this is a right triangle with two equal 45 degree angles—so this is an isosceles right triangle! The area of this shape equals quadrilateral ABCD (as we have only re-arranged the areas). So the other leg of this triangle AC also has a length of 6, and its area will be: 6 × 6 / 2 = 18

Amazing! This is unlike any math problem I've seen in class, a competition, or on a test. It's quite a neat trick!

Comment. Actually, my solution is basically the same as this only without flipping the triangle.

Thanks to Nestor Abad!

He figured out how the trigonometric expression simplifies to 18. Perhaps this is a good exercise for trigonometry students:

To simplify the expression

$$\frac{6(6\tan 40^\circ)}{2} + \frac{\left(\frac{6}{\cos 40^\circ}\frac{\sin 5^\circ}{\sin 45^\circ}\right)\left(\frac{6}{\cos 40^\circ}\right)\sin 130^\circ}{2}$$

first we can factor out the quantity $\frac{6\cdot 6}{2} = 18$ that appears in both sums and after arranging the fractions we get

$$18\left(\tan 40^\circ + \frac{\sin 5^\circ\ \sin 130^\circ}{\cos^2 40^\circ\ \sin 45^\circ}\right).$$

Now, $\sin 130^\circ = \sin 50^\circ = \cos 40^\circ$ because of the identities

$$\sin(\alpha) = \sin(180^\circ - \alpha) \quad \text{and} \quad \sin(\alpha) = \cos(90^\circ - \alpha),$$

so we can get rid of $\sin 130^\circ$ and one of the $\cos 40^\circ$, getting

$$18\left(\tan 40^\circ + \frac{\sin 5^\circ}{\cos 40^\circ\ \sin 45^\circ}\right).$$

Now we use the identity for the sine of a difference:

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta,$$

so we can write

$$\sin 5^\circ = \sin(45^\circ - 40^\circ) = \sin 45^\circ\cos 40^\circ - \cos 45^\circ\sin 40^\circ,$$

obtaining

$$18\left(\tan 40^\circ + \frac{\sin 45^\circ\cos 40^\circ - \cos 45^\circ\sin 40^\circ}{\cos 40^\circ\ \sin 45^\circ}\right).$$

Finally, splitting this fraction into two leads us quickly to the final answer after some simplifications:

$$18\left(\tan 40^\circ + \frac{\cancel{\sin 45^\circ\cos 40^\circ}}{\cancel{\cos 40^\circ\ \sin 45^\circ}} - \frac{\cancel{\cos 45^\circ}\sin 40^\circ}{\cos 40^\circ\ \cancel{\sin 45^\circ}}\right) = 18(\cancel{\tan 40^\circ} + 1 - \cancel{\tan 40^\circ}) = \boxed{18}.$$

https://drive.google.com/file/d/1Ll1SfK94N9mgqPRzxaNGGyv9u-YTAHVJ/view?usp=sharing