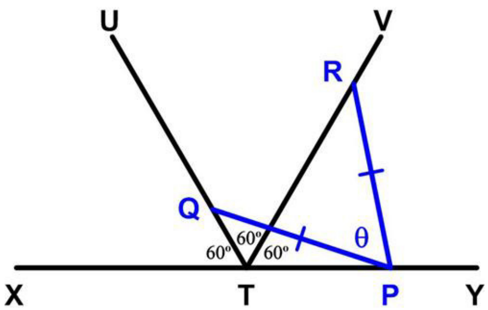
Ubiquitous 60 Degree Problem

3 April 2024

Jim Stevenson



This is an interesting problem from the Canadian Mathematical Society's 2001 *Olymon* ([1]).

Suppose that *XTY* is a straight line and that *TU* and *TV* are two rays emanating from *T* for which *XTU* = *UTV* = *VTY* = 60º. Suppose that *P*, *Q* and *R* are respective points on the rays *TY, TU* and *TV* for which *PQ* = *PR*. Prove that *QPR* = 60º.

My Solution

As shown in Figure 1, add a circle to the original figure centered on P and passing through Q and R. Flip the original diagram about the XY line. Since the original angles were 60°, so will be the mirrored versions. Therefore, the line UT extended to the lower half of the circle will be a straight line, since 60° + 60° + 60° = 180°. Similarly, the line RT extended to the lower half of the circle will also be a straight line. As shown in Figure 2, the angle at the intersection will be the sum of the corresponding inscribed angles or the average of the related central angles. Therefore,

$$60° = (\theta + \theta)/2 = \theta.$$

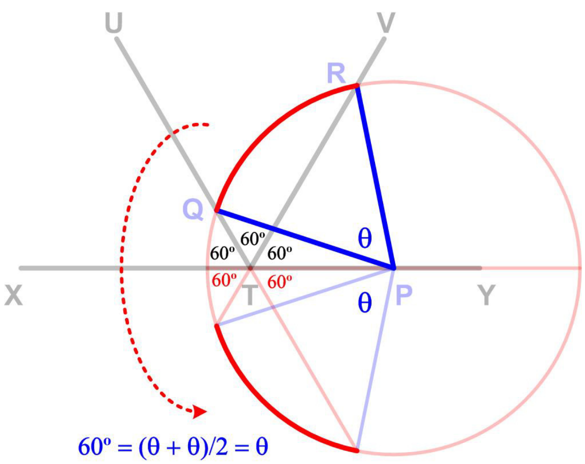


Figure 1

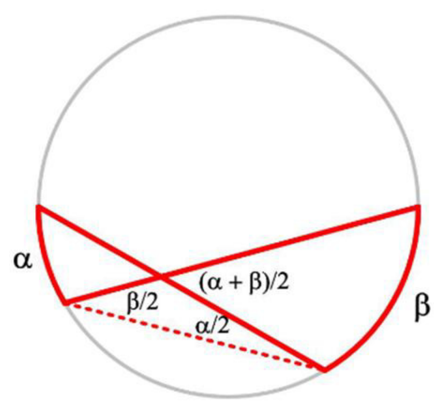


Figure 2

Olymon Solutions

I confess I did not check these solutions. And ***Solution 6*** was missing from the original.

Solution 1. Let $\mathfrak{R}$ be a rotation of 60° about T that takes the ray TU to TV. Then, if $\mathfrak{R}$ transforms $Q \to Q'$ and $P \to P'$, then Q' lies on TV and the line $Q'P'$ makes an angle of 60° with QP. Because of the rotation, $\angle P'TP = 60°$ and $TP' = TP$, whence $TP'P$ is an equilateral triangle.

Since $\angle Q'TP = \angle TPP' = 60°$, $TV \| P'P$. Let $\mathfrak{T}$ be the translation that takes P' to P. It takes Q' to a point Q'' on the ray TV, and $PQ'' = P'Q' = PQ$. Hence Q'' can be none other than the point R [why?], and the result follows.

Solution 2. The reflection in the line XY takes $P \to P$, $Q \to Q'$ and $R \to R'$. Triangles PQR' and $PQ'R$ are congruent and isosceles, so that $\angle TQP = \angle TQ'P = \angle TRP$ (since $PQ' = PR$). Hence $TQRP$ is a concyclic quadrilateral, whence $\angle QPR = \angle QTR = 60°$.

Solution 3. [S. Niu] Let S be a point on TU for which $SR \| XY$; observe that ΔRST is equilateral. We first show that Q lies between S and T. For, if S were between Q and T, then $\angle PSQ$ would be obtuse and $PQ > PS > PR$ (since $\angle PRS > 60° > \angle PSR$ in ΔPRS), a contradiction.

The rotation of $60°$ with centre R that takes S onto T takes ray RQ onto a ray through R that intersects TY in M. Consider triangles RSQ and RTM. Since $\angle RST = \angle RTM = 60°$, $\angle SRQ = 60° - \angle QRT = \angle TRM$ and $SR = TR$, we have that $\Delta RSQ \equiv \Delta RTM$ and $RQ = RM$. (ASA) Since $\angle QRM = 60°$, ΔRQM is equilateral and $RM = RQ$. Hence M and P are both equidistant from Q and R, and so at the intersection of TY and the right bisector of QR. Thus, $M = P$ and the result follows.

Solution 4. [H. Pan] Let Q' and R' be the respective reflections of Q and R with respect to the axis XY. Since $\angle RTR' = 120°$ and $TR = TR'$, $\angle QR'R = \angle TR'R = 30°$. Since Q, R, Q', R', lie on a circle with centre P, $\angle QPR = 2\angle QR'R = 60°$, as desired.

Solution 5. [R. Barrington Leigh] Let W be a point on TV such that $\angle WPQ = 60° = \angle WTU$. [Why does such a point W exist?] Then $WQTP$ is a concyclic quadrilateral so that $\angle QWP = 180° - \angle QTP = 60°$ and ΔPWQ is equilateral. Hence $PW = PQ = PR$.

Suppose $W \neq R$. If R is farther away from T than W, then $\angle RPT > \angle WPT > \angle WPQ = 60° \Rightarrow 60° > \angle TRP = \angle RWP > 60°$, a contradiction. If W is farther away from T than R, then $\angle WPT > \angle WPQ = 60° \Rightarrow 60° > \angle RWP = \angle WRP > 60°$, again a contradiction. So $R = W$ and the result follows.

Solution 7. [P. Cheng] Determine S on TU and Z on TY for which $SR \| XY$ and $\angle QRZ = 60°$. Observe that $\angle TSR = \angle SRT = 60°$ and $SR = RT$.

Consider triangles SRQ and TRZ. $\angle SRQ = \angle SRT - \angle QRT = \angle QRZ - \angle QRT = \angle TRZ$; $\angle QSR = 60° = \angle ZTR$, so that $\Delta SRQ = \Delta TRZ$ (ASA).

Hence $RZ = RQ \Rightarrow \Delta RQZ$ is equilateral $\Rightarrow RZ = ZQ$ and $\angle RZQ = 60°$. Now, both P and Z lie on the intersection of TY and the right bisector of QR, so they must coincide: $P = Z$. The result follows.

Solution 8. Let the perpendicular, produced, from Q to XY meet VT, produced, in S. Then $\angle XTS = \angle VTY = 60° = \angle XTU$, from which is can be deduced that TX right bisects QS. Hence $PS = PQ = PR$, so that Q, R, S are all on the same circle with centre P.

Since $\angle QTS = 120°$, we have that $\angle SQT = \angle QSR = 30°$, so that QR must subtend an angle of $60°$ at the centre P of the circle. The desired result follows.

Solution 9. [A.Siu] Let the right bisector of QR meet the circumcircle of TQR on the same side of QR at T in S. Since $\angle QSR = \angle QTR = 60°$ and $QS = QR$, $\angle SQR = \angle SRQ = 60°$. Hence $\angle STQ = 180° - \angle SRQ = 120°$. But $\angle YTQ = 120°$, so S must lie on TY. It follows that $S = P$.

Solution 10. Assign coordinates with the origin at T and the x−axis along XY. The the respective coordinates of Q and R have the form $(u, -\sqrt{3}u)$ and $(v, \sqrt{3}v)$ for some real u and v. Let the coordinates of P be $(w, 0)$. Then $PQ = PR$ yields that $w = 2(u + v)$. [*Exercise:* work it out.]

$$\begin{aligned}|PQ|^2 - |QR|^2 &= (u-w)^2 + 3u^2 - (u-v)^2 - 3(u+v)^2 \\ &= w^2 - 2uw - 4v(u+v) = w^2 - 2uw - 2vw \\ &= w^2 - 2(u+v)w = 0 \ .\end{aligned}$$

Hence $PQ = QR = PR$ and ΔPQR is equilateral. Therefore $\angle QPR = 60°$.

Solution 11. [J.Y. Jin] Let $\mathfrak{C}$ be the circumcircle of ΔPQR. If T lies strictly inside $\mathfrak{C}$, then $60° = \angle QTR > \angle QPR$ and $60° = \angle PTR > \angle PQR = \angle PRQ$. Thus, all three angle of ΔPQR would be less than $60°$, which is not possible. Similarly, if T lies strictly outside $\mathfrak{C}$, then $60° = \angle QTR < \angle QPR$ and $60° = \angle PTR < \angle PQR = \angle PRQ$, so that all three angles of ΔPQR would exceed $60°$, again not possible. Thus T must be on $\mathfrak{C}$, whence $\angle QPR = \angle QTR = 60°$.

Solution 12. [C. Lau] By the Sine Law,

$$\frac{\sin \angle TQP}{|TP|} = \frac{\sin 120°}{|PQ|} = \frac{\sin 60°}{|PR|} = \frac{\sin \angle TRP}{|TP|} \ ,$$

whence $\sin \angle TQP = \sin \angle TRP$. Since $\angle QTP$, in triangle QTP is obtuse, $\angle TQP$ is acute.

Suppose, if possible, that $\angle TRP$ is obtuse. Then, in triangle TPR, TP would be the longest side, so $PR < TP$. But in triangle TQP, PQ is the longest side, so $PQ > TP$, and so $PQ \neq PR$, contrary to hypothesis. Hence $\angle TRP$ is acute. Therefore, $\angle TQP = \angle TRP$. Let PQ and RT intersect in Z. Then, $60° = \angle QTZ = 180° - \angle TQP - \angle QZT = 180° - \angle TRP - \angle RZP = \angle QPR$, as desired.

References

[1] Barbeau, Edward, Dragos Hrimiuk, and Valeria Pandeleva, "Problem 65", *Olymon*, Vol. 2, 2001. (http://www.math.utoronto.ca/barbeau/olymon2001.pdf) "Olymon (the Mathematics Olympiads Correspondence Program) is a problems correspondence program for secondary students sponsored jointly by the Canadian Mathematical Society and the Mathematics Department of the University of Toronto. Currently, a Problem of the Week is posed and secondary students are invited to submit solutions for grading. It is intended to provide students with some background mathematics and experience and feedback on solving and writing up solutions for problems."