Double Areas Puzzles

31 December 2024

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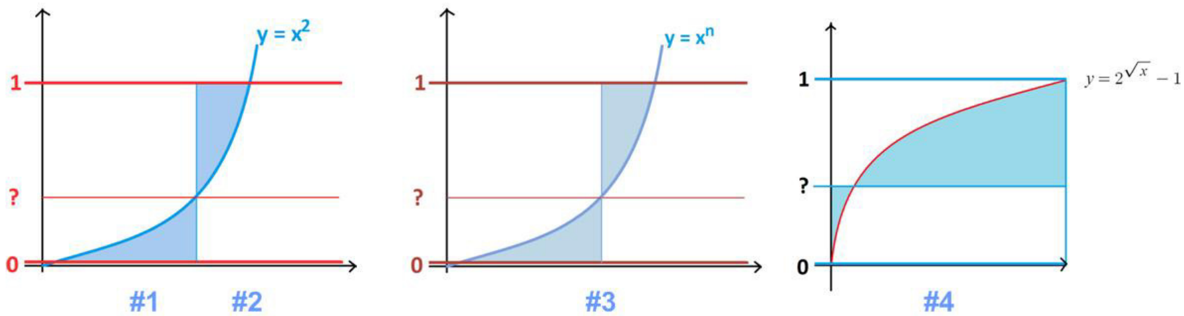

A while ago James Tanton provided a series of puzzles:

Puzzle #1[1] At what value between 0 and 1 does a horizontal line at that height produce two regions of equal area as shown on the graph of $y = x^2$?

Puzzle #2[2] A horizontal line is drawn between the lines $y = 0$ and $y = 1$, dividing the graph of $y = x^2$ into two regions as shown. At what height should that line be drawn so that the sum of the areas of these two regions is minimal?

Puzzle #3[3] A horizontal line is drawn between the lines y = 0 and y = 1, dividing the graph of $y = x^n$ into two regions as shown ($n > 0$). At what height should that line be drawn so that the sum of the areas of these two regions is minimal? Does that height depend on the value of n?

Puzzle #4[4] What horizontal line drawn between $y = 0$ and $y = 1$ on the graph of $y = 2^{\sqrt{x}} - 1$ minimizes the sum of the two shaded areas shown?

Solution Puzzle #1

The area below the line $B(x)$ is given by

$$B(x) = \int_0^x t^2 dt$$

and the area above the line $U(x)$ is given by

$$U(x) = \int_x^1 (1 - t^2) dt$$

So we are interested in finding the x such that $B(x) = U(x)$, that is,

$$\int_0^x t^2 dt = \int_x^1 (1 - t^2) dt = \int_x^1 dt - \int_x^1 t^2 dt$$

So

1 23 December 2022 (https://twitter.com/jamestanton/status/1606239945924001793)
2 26 December 2022 (https://twitter.com/jamestanton/status/1607327864529059843)
3 27 December 2022 (https://twitter.com/jamestanton/status/1607679053762371585)
4 28 December 2022 (https://twitter.com/jamestanton/status/1608046851864788992)

$$\left.\frac{t^3}{3}\right|_0^1 = \int_0^1 t^2 dt = \int_x^1 dt = 1 - x$$

$$x = 1 - {}^1/_3 = {}^2/_3$$

Thus the horizontal line is at $y = ({}^2/_3)^2 = {}^4/_9$

Solution Puzzle #2

Now we are interested in minimizing the sum $A(x) = B(x) + U(x)$ of the lower and upper areas, that is,

$$A(x) = \int_0^x t^2 dt + \int_x^1 (1-t^2) dt$$

Taking the derivative (via a form of the Fundamental Theorem of Calculus and using $\int_b^a \; = -\int_a^b$) and setting it to zero, we get

$$A'(x) = x^2 - (1-x^2) = 2x^2 - 1 = 0 \;\Rightarrow\; x = 1/\surd 2$$

$$A''(x) = 4x > 0 \text{ since } x > 0$$

Therefore, $x = 1/\surd 2$ is a minimum. And $y = (1/\surd 2)^2 = {}^1/_2$.

Solution Puzzle #3

Rather than consider just this generalization, suppose we have a general, positive, differentiable function $y = f(x)$ (and so continuous). Then the area $A(x) = B(x) + U(x)$ becomes

$$A(x) = \int_0^x f(t)dt + \int_x^1 (1 - f(t))dt$$

Differentiating, we get $A'(x) = f(x) + (1 - f(x)) = 2f(x) - 1 = 0$

So $y = f(x) = ½$.

Now, $A''(x) = 2f'(x)$

So if f is strictly increasing, then $f'(x) > 0$ implies $A''(x) > 0$, and ½ is a minimum point for $A(x)$.

Comment. Problems #2-#3 are more or less the same as problem 2 of Nakul Dawra's coffin problems.[5] Only there Dawra considered a general, monotonically increasing function and not just $y = x^n$ or a strictly increasing differentiable function $y = f(x)$.

Solution Puzzle #4

Tanton's complete statement for this puzzle included this:

[The answer to all these questions is $y = ½$, if we're dealing with a strictly increasing function. The real question is how to best explain this.]

This is true for Puzzles #2-#3, but not this one. However, in Tanton's solution to this puzzle, where he sets $y = f(x) = 2^{\sqrt{x}} - 1$, so $f(0) = 0$ and $f(1) = 1$, he correctly says:

[5] "Three Coffin Problems" (Post: https://josmfs.net/2019/01/19/three-coffin-problems/, Article: http://josmfs.net/wordpress/wp-content/uploads/2019/01/Three-Coffin-Problems-180907.pdf)

In this case the answer is actually $y = f(½)$, not $y = ½$. It's got to do with the fact that the derivative of the integral is the function itself. In this case, in particular, the derivative with respect to y is the left horizontal blue line minus the right horizontal blue line.

So I am not sure why Tanton made the initial statement about $y = ½$. Alternatively, Tanton could have made a misprint and intended the areas to be vertical rather than horizontal, but then the second statement wouldn't be true.

Another way to see the correct solution to the given problem is just interchange x and y (flip around the 45° line) to reduce the problem to the previous one (Figure 1). This is equivalent to considering the inverse function to f.

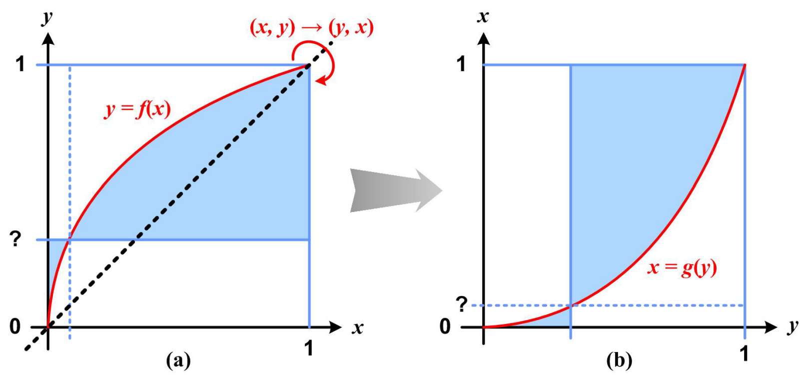

Figure 1

We can use the inverse function explicitly to solve the problem "directly" as follows (Figure 2). Let g be the inverse function of f, that is, $x = g(f(x))$. The sum of the two areas satisfies the scheme

$$A = \int_0^y x\,dy + \int_y^1 (1-x)\,dy$$

or specifically

$$A(x) = \int_0^{f(x)} g(t)\,dt + \int_{f(x)}^1 (1-g(t))\,dt\ ,$$

which is really the previous solutions for Puzzles #2-#3 applied to the graph in Figure 1(b). Taking derivatives yields

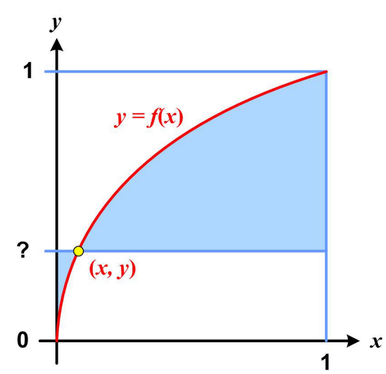

Figure 2

$$A'(x) = g(f(x))f'(x) - (1-g(f(x)))f'(x) = (2x-1)f'(x) = 0$$

when $x = ½$, since $f'(x) > 0$. Therefore, the horizontal line is at $y = f(½)$.