Sphere and Plane Puzzle

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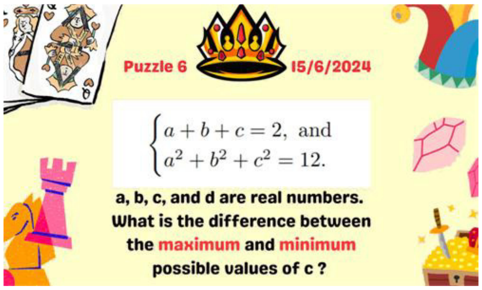
Jim Stevenson



This is another puzzle from BL's Weekly Math Games.[1]

$$a + b + c = 2, \text{ and}$$
$$a^2 + b^2 + c^2 = 12$$

where *a, b,* and *c* are real numbers. What is the difference between the maximum and minimum possible values of *c*?

The original problem statement mentioned a fourth real number *d*, but I considered it a typo, since it was not involved in the problem.

Solution

I first approached this problem geometrically, that is, I realized we were considering the locus of points (*a, b, c*) lying on the intersection of the plane

$$a + b + c = 2 \qquad (1)$$

with the sphere

$$a^2 + b^2 + c^2 = 12, \qquad (2)$$

which would be a tilted circle (Figure 1).

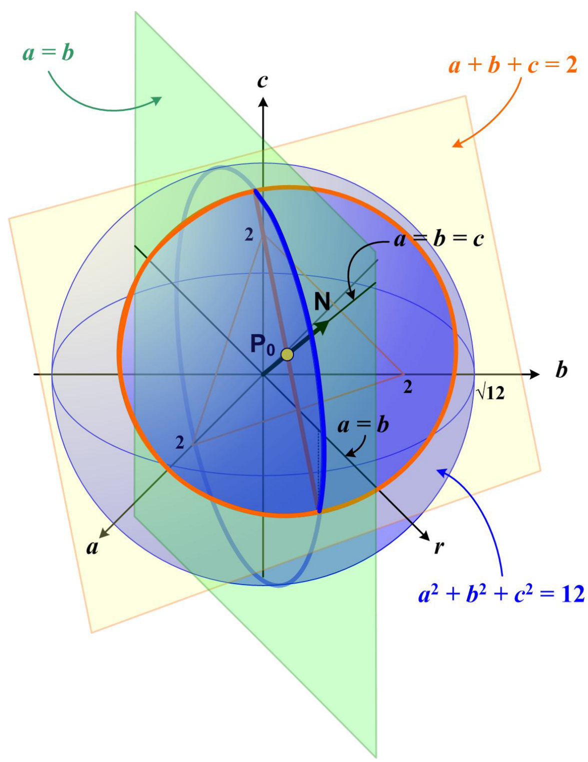


Figure 1

Recall equation (1) defines a plane since if we define vectors $\mathbf{N} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{P_0} = ⅔\,\mathbf{i} + ⅔\,\mathbf{j} + ⅔\,\mathbf{k}$, and $\mathbf{P} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$, then

$$\mathbf{N}\cdot(\mathbf{P} - \mathbf{P_0}) = 0 \iff \mathbf{N}\cdot\mathbf{P} = \mathbf{N}\cdot\mathbf{P_0}$$
$$\iff a + b + c = 2.$$

So the vector $\mathbf{P} - \mathbf{P_0}$ sweeps out all points (*a, b, c*) lying in a plane perpendicular to **N** and cutting the *a*-, *b*-¸*c*-axes at 2.

From Figure 1 we see that the symmetry in the equations between *a* and *b* (swapping *a* and *b* does not change the values) means the value of *c* at a point (*a, b*) is the same as at (*b, a*). That is, *c* is the same on either side of the 45° line in the *ab*-plane. Therefore, the max and min of *c* must occur when $a = b$, that is, where the vertical plane $a = b$ through the *c*-axis and this 45° line cuts the tilted circle.

Parmeterize the line $a = b$ with *r*, where $r^2 = a^2 + b^2 = 2a^2$, so that $r = \sqrt{2}\,a$. Then $a = r/\sqrt{2}$ and equations (1) and (2) become

[1] 15 June 2024 (https://medium.com/bellas-weekly-math-games/whats-the-difference-between-max-and-min-6857f13001b5)

$$c = 2 - \sqrt{2}\, r \quad (3)$$

$$c^2 + r^2 = 12 \quad (4)$$

(See Figure 2.) Solving for r in equation (3) and substituting the result in equation (4) yields

$$3\,c^2 - 4\,c - 20 = 0.$$

The quadratic formula gives

$$c = {}^{10}/_3,\ -2,$$

or $c\ \text{max} - c\ \text{min} = {}^{16}/_3 = 5⅓$.

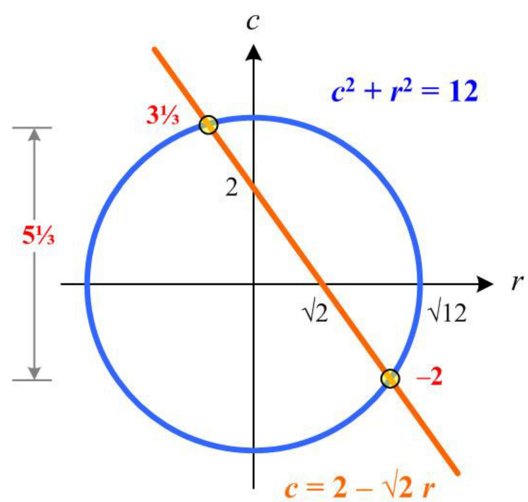


Figure 2

Analytic Approach

If the geometric argument is a bit unpersuasive, I considered some calculus. Equation (1) means that c is defined implicitly as a function of a and b. Equation (2) provides a second such implicit definition for c. But for the two definitions to simultaneously hold for c means b, say, is implicitly defined as a function of a. So the points (a, b, c) are defined by the variation of a single parameter a, and thus form a curve.

Taking derivatives implicitly with respect to a in the two equations yields two linear equations in the derivatives:

$$1 + \frac{db}{da} + \frac{dc}{da} = 0$$

$$2a + 2b\frac{db}{da} + 2c\frac{dc}{da} = 0$$

Eliminating db/da from the two equations yields

$$(b - c)\frac{dc}{da} = a - b$$

Therefore, at the critical points where $dc/da = 0$, again we have $a = b$, and the solution above yields

$$c\ \text{max} - c\ \text{min} = 5⅓.$$

(Again, I do not subscribe to BL's website, so I don't know what the solution given there was.)