Mystery Quadratic

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Presh Talwalkar has an interesting new problem.[1]

Students and teachers found a recent test in New Zealand to be confusing and challenging for covering topics that were not taught in class.[2]

For the equation below, find the value of k for which the equation has numerically equal but opposite signs (for example, 2 and –2):

$$\frac{x^2-2x}{4x-1}=\frac{k-1}{k+1}$$

The problem didn't mention how old the students were, but the solution to another problem on the test indicates they needed to know calculus.

My Solution

Cross multiplying and simplifying the original equation yields

$$x^2-\frac{6k-2}{k+1}x+\frac{k-1}{k+1}=0 \tag{1}$$

By a version of the Fundamental Theorem of Algebra[3] this equation must in general have two complex roots. That is, if α and β are the roots, then the polynomial must factor as

$$(x-\alpha)\,(x-\beta)=x^2-(\alpha+\beta)x+\alpha\beta$$

We are assuming for the problem that $\beta = -\alpha$, which means the coefficient of x must be 0. The value of k that will make that happen is $k = ⅓$.

In fact, we can say more. We can find out what the roots are. We could use the quadratic formula, which I did at first, but now we know the value of the constant term in equation (1) is –½. If the roots are complex and not real, then, since the coefficients are real, the roots must be complex conjugates. That is, $\beta = \bar{\alpha}$. But $\bar{\alpha} = -\alpha$ means $a - ib = -a - ib$ and that can happen only if the real part $a = 0$. But

$$\alpha\beta = -\alpha^2 = -(ib)^2 = b^2 = -½$$

which is impossible, since b is a real number. Therefore the roots must be real, and so $b = 0$ and

$$\alpha\beta = -a^2 = -½$$

means the roots are $1/\sqrt{2}$ and $-1/\sqrt{2}$.

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[1] 1 December 2024 (https://mindyourdecisions.com/blog/2024/12/01/new-zealand-test-leaves-students-in-tears/)

[2] Always a dangerous move in a timed test for students fresh to the material. It may pick out the one or two exceptional students, but it will ruin any finer gradation in class evaluation, since it may cause panic and cause the students to give less attention to the problems they could solve.

[3] https://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra

Talwalkar Solution

Talwalkar's solution begins the same as mine.

$$(x^2 - 2x)/(4x - 1) = (k - 1)/(k + 1)$$

$$(x^2 - 2x)(k + 1) = (4x - 1)(k - 1)$$

$$x^2(k + 1) - 2x(k + 1) = 4x(k - 1) - (k - 1)$$

$$x^2(k + 1) + x(-2k - 2) = x(4k - 4) - (k - 1)$$

$$x^2(k + 1) - x(-6k + 2) + (k - 1) = 0$$

Talwalkar assumes without proof that the solutions to the equation must be real and sort of elides the fact that the leading coefficient in the equation is not 1.

If an equation has roots of opposite signs, like r and $-r$, then it must be a difference of squares because:

$$(x - r)(x + r) = x^2 - r^2$$

Thus the coefficient of x is 0. So we must have the coefficient $-6k + 2$ be equal to 0.

$$-6k + 2 = 0$$

$$k = 1/3$$