Tricky Ratio Puzzle

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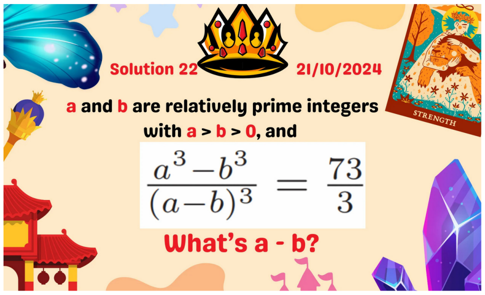

This is an interesting algebra problem[1] from *BL's Weekly Math Games*, which is behind a subscription wall.

If a and b are relatively prime integers with $a > b > 0$, and

$$(a^3 - b^3)/(a - b)^3 = 73/3,$$

what is $a - b$?

In fact, it is possible to solve for a and b individually as well.

Solution

First, notice that $f(a, b) = a^3 - b^3$ and $g(a, b) = (a - b)^3$ are homogeneous of degree 3, that is, $f(ta, tb) = t^3 f(a, b)$ and $g(ta, tb) = t^3 g(a, b)$. Therefore $f(ta, tb)/g(ta, tb) = f(a, b)/g(a, b)$. So all solutions are unique up to a multiplicative constant t. But since a and b are assumed to be relatively prime, we are only interested in $t = 1$.

Since $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$, the problem becomes

$$(a^2 + ab + b^2)/(a - b)^2 = 73/3$$

Now $a^2 + ab + b^2 = (a - b)^2 + 3ab$ so

$$3ab/(a - b)^2 = 73/3 - 1 = 70/3$$

Therefore

$$(a - b)^2 = 9ab/70.$$

This means $ab/70$ is a perfect square k^2 or $ab = k^2\,70$ for some integer k. And that would imply $a - b = 3k$ for some integer k.

Solving for *a* and *b*. Let $k > 0$ be any integer. Then k has a unique factorization into products of, possibly non-distinct, primes, $k = u\, p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot \ldots \cdot p_n$ where u is unit 1, which we will ignore. The duplicate primes in k^2 can be paired together so that k^2 becomes a product of prime squares. Since a and b are relatively prime, all the (necessarily even) powers of a given prime factor must either divide a or divide b. Therefore $ab = k^2\,70 = p^2q^2\,70$ for various relatively prime p and q (no prime factors in common), where p^2 divides a and q^2 divides b. So $a = p^2 70/c$ and $b = q^2 c$, where c is any factor of 70, including 1 and 70.

Then $a = p^2 70/c$, $b = q^2 c$, $ab = p^2q^2\,70$, and $a - b = 3pq$, yields

$$p^2 70/c - cq^2 = 3pq \quad \text{or} \quad \mathrm{A}p^2 + \mathrm{B}pq + \mathrm{C}q^2 = 0$$

where $\mathrm{A} = 70/c$, $\mathrm{B} = -3$, and $\mathrm{C} = -c$. Let $r = p/q$. then we have

$$\mathrm{A}r^2 + \mathrm{B}r + \mathrm{C} = 0.$$

Solving for r via the quadratic formula with $\mathrm{B}^2 - 4\mathrm{AC} = 9 + 4\cdot(70/c)\cdot c = 289 = 17^2$ yields

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[1] 26 October 2024 (https://medium.com/bellas-weekly-math-games/an-algebra-challenge-0a0123be470c)

$$r = (3 + 17)/2A \text{ or } r = c/7 = p/q$$

Therefore,

$$a = p^2 70/c = p^2(q/p)10 = 10\,pq$$

$$b = q^2 c = q^2(7p/q) = 7\,pq$$

But a and b are relatively prime, so $k = pq = 1$.

Therefore there are no other solutions for a and b than $k = 1$. So $a = 10$ and $b = 7$ and $a - b = 3$.

(I don't know what BL's solution is, since I don't subscribe.)