# Spy Gift Problem Rigorous Solution

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The similarity of the "Umbrella Problem"[1] with the "Spy Gift Problem"[2] suggests the permutation approach in the Umbrella Problem solution would work for the Spy Gift Problem as well. And that would explain why Alex Bellos's circle form of the solution works, since it is just a visual representation of a permutation cycle.

The Secret Santa requirement is just the permutation $p$ on 9 spies where $p$ assigns to each spy the spy they give a present to, and where

$$p^2(n) = n + 1, \text{ for } n = 001, 002, \ldots, 008 \text{ and } p^2(009) = 001$$

This means the permutation $p$ cannot have any 1-cycles or 2-cycles, since $p(n) \neq n$ (no one gives a gift to themselves) and $p^2(n) \neq n$, for any $n$.

If there were a 3-cycle, then for some $k$, $k = p^3(k)$, that is, $p = (k, *, *)(*, \ldots, *)$ using the cycle notation from the Umbrella Problem with an asterisk representing an unknown integer value. But $p^2(k) = k + 1$. So $p = (k, *, k + 1)(*, \ldots, *)$ or $k = p^3(k) = p(p^2(k)) = p(k + 1)$. But this implies $p(k) = p^2(k + 1) = k + 2$ or $p = (k, k + 2, k + 1)(*, \ldots, *)$, which means $p^2(k + 2) = k \neq k + 3$, a contradiction.

And if there were a 4-cycle, then for some $k$, $k = p^4(k) = p^2(p^2(k)) = p^2(k + 1) = k + 2$, another contradiction.

This means there can't be any 5-, 6-, 7-, or 8-cycles, since that would mean there would have been a 4-, 3-, 2-, or 1-cycle. So the only cycle we can have is a 9-cycle permutation. What this argument boils down to is that if there were any sub-cycles in the permutation, the requirement $p^2(n) = n + 1$, including the wraparound, would not hold for all $n = 1, 2, \ldots, 9$.

Using the asterisk cycle notation above, $p^2(n) = n + 1$ with wraparound means

$$p = (001, *, 002, *, 003, *, 004, *, 005)$$

So 005 goes to 001 after one application of $p$. Therefore, a second application should take the result to 006, that is, 001 should go to 006. And thus we fill in the * with the remaining successive numbers:

$$p = (001, 006, 002, 007, 003, 008, 004, 009, 005),$$

which is just what Alex Bellos's "filling in the circle" does. (But Bellos did not show there was *only* a 9-cycle for the permutation.)

The clues were all there in the "Spy Gift Problem", especially the circle solution of Alex Bellos, that the problem should be solved with permutations, and that the permutations should be interpreted via cycles. I just didn't see it.

[1] https://josmfs.net/2024/09/07/the-umbrella-problem/

[2] https://josmfs.net/2024/02/24/spy-gift-problem/