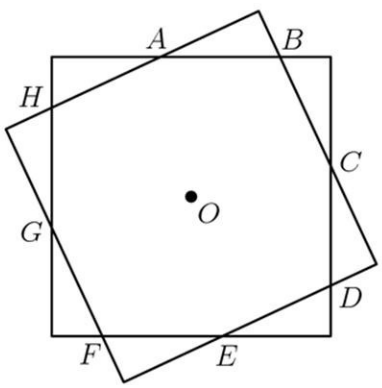
Another Octagonal Area Problem

19 June 2024

Jim Stevenson



This is a fairly straight-forward problem from the 1999 AIME problems ([1]).

The two squares shown share the same center O and have sides of length 1. The length of AB is 43/99 and the area of octagon ABCDEFGH is m/n where m and n are relatively prime positive integers. Find m + n.

My Solution

Notice that the identical squares having the same center means they are rotated versions of each other. Figure 1 shows the circumscribing circle with the vertices of the squares lying on it. The octagonal area of interest is shaded in blue. We want to show the 8 right triangles created by the rotated square are all congruent. Consider the two shaded in green. They share one vertex angle and so the third angles must be identical. Thus the right triangles are similar. Now the red triangle, with sides the radius of the circle, is isosceles and so has identical base angles. But the radii make angles of 45° at the corners of the squares. So the yellow triangle must have identical base angles as well and be isosceles. This means the two similar green triangles have a side in common, and so are congruent. By symmetry, this argument works for all the other right triangles and so they are all congruent.

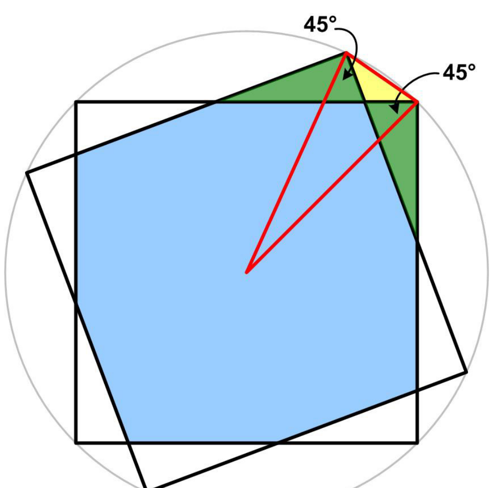


Figure 1

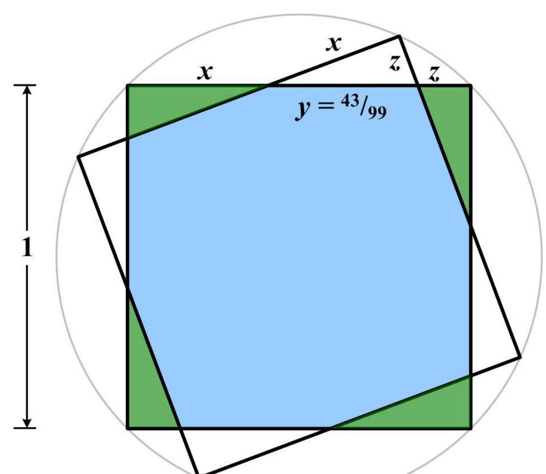


Figure 2

Label the sides of the green triangles x, y, and z as shown in Figure 2. Then the area of the octagon is given by

$$1 - 4(\tfrac{1}{2}\,xz) = 1 - 2xy$$

Now we have the following relations for $y = {}^{43}/_{99}$

$$x + y + z = 1 \;\Rightarrow\; x + z = 1 - {}^{43}/_{99} = {}^{56}/_{99} \text{ and}$$

$$x^2 + z^2 = y^2 = ({}^{43}/_{99})^2$$

But

$$({}^{56}/_{99})^2 = (x + z)^2 = x^2 + z^2 + 2xy = ({}^{43}/_{99})^2 + 2xy$$

So

$$2xy = (56^2 - 43^2)/99^2 = (56 - 43)(56 + 43)/99^2 = 13 \cdot 99/99^2 = {}^{13}/_{99}$$

Therefore the area of the octagon is

$$1 - {}^{13}/_{99} = {}^{86}/_{99}$$

Since 86 and 99 are relatively prime, the answer is

$$86 + 99 = 185$$

AIME Solutions

Solution 1

Triangles *AOB, BOC, COD*, etc. [blue triangles (Figure 3)] are congruent by symmetry (you can prove it rigorously by using the power of a point to argue that exactly two chords of length 1 in the circumcircle of the squares pass through *B*, etc.),[1] and each area is $({}^{43}/_{99} \cdot {}^{1}/_{2})/2$. Since the area of a triangle is $bh/2$, the area of all 8 of them is ${}^{86}/_{99}$ and the answer is 185.

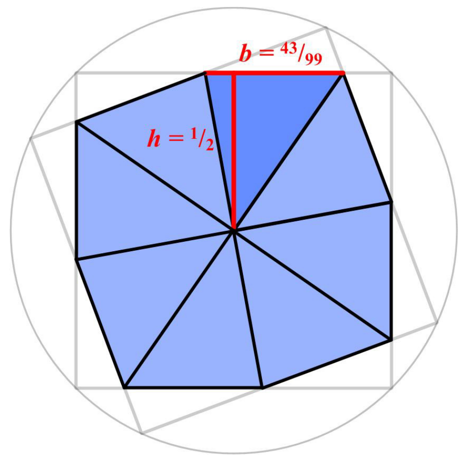


Figure 3

Solution 2

This is the same as my solution only AIME doesn't explicitly prove the 8 right triangles are congruent.

References

[1] "Problem 4" 1999 AIME Problems (https://artofproblemsolving.com/wiki/index.php/1999_AIME_Problems)

[1] JOS: I am afraid I don't follow this reasoning. But perhaps it doesn't matter. Using my argument for the 8 congruent small right triangles the bases of all the blue triangles are the same, namely, ${}^{43}/_{99}$, and the altitudes are the same by virtue of the squares, namely, ½.