The Umbrella Problem

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This is a rather mind-boggling problem from the 1947 *Eureka* magazine ([1]).

Six men, A, B, C, D, E, F, of negligible honesty, met on a perfectly rough day, each carrying a light inextensible umbrella. Each man brought his own umbrella, and took away—let us say "borrowed"—another's. The umbrella borrowed by A belonged to the borrower of B's umbrella. The owner of the umbrella borrowed by C borrowed the umbrella belonging to the borrower of D's umbrella. If the borrower of E's umbrella was not the owner of that borrowed by F, who borrowed A's umbrella?

My Solution

My solution is a clumsy trial-and-error approach, which happened to luck out on the first try. I laid out the choices on lines as shown in Figure 1. I joined one letter with another if the first borrowed the umbrella of the second.

Using the first sentence below, I arbitrarily chose C to be the borrower of B's umbrella and inserted its letter in red for the borrower. The second sentence meant that B had to be the owner of the umbrella borrowed by C. I arbitrarily assigned E as the borrower of D's umbrella. That meant that B was not the owner of the umbrella borrowed by F, since C had borrowed B's umbrella.

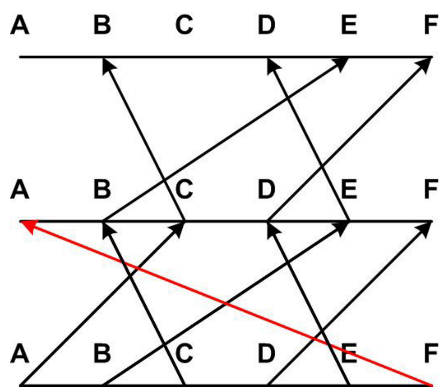

Figure 1

From the problem statement (with red letters added as described above):

1. The umbrella borrowed by A belonged to the borrower C of B's umbrella.
2. The owner B of the umbrella borrowed by C borrowed the umbrella belonging to the borrower E of D's umbrella.
3. The borrower B of E's umbrella was not the owner of that borrowed by F.
4. Then who borrowed A's umbrella?

Therefore, everything is consistent up to this point, so that leaves A as the owner of the umbrella borrowed by F.

I did not consider other possible solutions.

Eureka Solution

The *Eureka* solution is very elegant and finds all the solutions to the problem.

The simple fact is relevant, that any permutation p of the numbers 1, 2, ... , n is expressible as the product of cycles,

$$p = (\alpha, \beta, \gamma, \ldots \zeta)\ (\eta, \theta, \ldots)\ (\ldots, \omega)$$

Here, α, ..., ω represent the numbers from 1 to n in some arrangement, each number appearing only once. It is implied that p sends α into β, β into γ, ... , ζ into α, and η into θ, and so forth. The number of terms enclosed between a pair of brackets is the order of the corresponding cycle. Cycles of order 1 may of course be omitted.

In the umbrella problem, a permutation p of A, B, C, D, E, F is defined by the stipulation that the umbrella belonging to x was borrowed by px, where x runs through the letters A, ... , F. The data are now:

(i) p contains no cycle of order 1;

(ii) $A = p^2B$;

(iii) $C = p^3D$;

(iv) $p^2E \neq F$.

It follows from (i) that p must be of one of four forms [essentially all partitions of A, B, C, D, E, F, each involving more than one letter],

either (*, *) (*, *) (*, *) [$\Rightarrow p^2 = 1$ (identity permutation)]

or (*, *) (*, *, *, *)

or (*, *, *) (*, *, *)

or (*, *, *, *, *, *).

The first form is incompatible with (ii), which reduces the alternatives to

(*, *) (B, *, A, *) and (B, *, A) (*, *, *) and (B, *, A, *, *, *).

Using (iii) similarly, we obtain the alternatives

(C, D) (B, *, A, *) and (B, D, A, *, C, *) and (B, C, A, *, D, *).

Finally, (iv) shows that p must be either (B, D, A, F, C, E) or (B, C, A, F, D, E). Thus, pA = F; the borrower of A's umbrella was F.

Comment.

The similarity of this problem with the "**Spy Gift Problem**"[1] suggests this permutation approach would work for it as well. And that would explain why the circle form of the solution works, since it is just a visual representation of a permutation cycle.

The Secret Santa requirement is just the permutation p on 9 spies where p assigns to each spy the spy they give a present to, and where

$$p^2(n) = n + 1, \text{ for } n = 001, 002, \ldots, 008 \text{ and } p^2(009) = 001$$

This means the permutation p cannot have any 1-cycles or 2-cycles, since $p(n) \neq n$ (no one gives a gift to themselves) and $p^2(n) \neq n$, for any n.

If there were a 3-cycle, then for some k, $k = p^3(k)$, that is, $p = (k, *, *)(*, \ldots, *)$ using the asterisk notation above. But $p^2(k) = k + 1$. So $p = (k, *, k + 1)(*, \ldots, *)$ or $k = p^3(k) = p(p^2(k)) = p(k + 1)$.

[1] https://josmfs.net/2024/02/24/spy-gift-problem/

But this implies $p(k) = p^2(k + 1) = k + 2$ or $p = (k, k + 2 , k + 1)(*, \ldots, *)$, which means $p^2(k + 2) = k \neq k + 3$, a contradiction.

And if there were a 4-cycle, then for some k, $k = p^4(k) = p^2(p^2(k)) = p^2(k + 1) = k + 2$, another contradiction.

This means there can't be any 5-, 6-, 7-, or 8-cycles, since that would mean there would have been a 4-, 3-, 2-, or 1-cycle. So the only cycle we can have is a 9-cycle permutation. What this argument boils down to is that if there were any sub-cycles in the permutation, the requirement $p^2(n) = n + 1$, including the wraparound, would not hold for all $n = 1, 2, \ldots, 9$.

Using the asterisk cycle notation above, $p^2(n) = n + 1$ with wraparound means

$$p = (001, *, 002, *, 003, *, 004, *, 005)$$

So 005 goes to 001 after one application of p. Therefore, a second application should take the result to 006, that is, 001 should go to 006. And thus we fill in the * with the remaining successive numbers:

$$p = (001, 006, 002, 007, 003, 008, 004, 009, 005),$$

which is just what Alex Bellos's "filling in the circle" does. (But Bellos did not show there was *only* a 9-cycle for the permutation.)

The clues were all there in the "Spy Gift Problem", especially the circle solution of Alex Bellos, that the problem should be solved with permutations, and that the permutations should be interpreted via cycles. I just didn't see it.

References

[1] "The Umbrella Problem", *Eureka*, The Journal of the Archimedeans, The Cambridge University Mathematical Society: Junior Branch of the Mathematical Association, No. 9, April 1947. p.22 (https://www.archim.org.uk/eureka/archive/Eureka-9.pdf)