Serious Series
17 December 2019
Jim Stevenson

On searches a 'calculus-free' method to evaluate : $S_2 = \sum_{k=1}^{\infty} \frac{k(k+1)}{2^k} \left(= \text{ also } \sum_{k=0}^{\infty} \frac{k(k+1)}{2^k} \right)$

We know, $S_0 = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$

then , $S_1 = \sum_{k=1}^{\infty} \frac{k}{2^k} = \sum_{k=1}^{\infty} \frac{(k-1+1)}{2^k} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(k-1)}{2^{k-1}} + \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{S_1}{2} + S_0$

$$\implies S_1 = 2$$

$$S_2 = \sum_{k=1}^{\infty} \frac{k(k+1)}{2^k} = \sum_{k=1}^{\infty} \frac{(k-1)k + 2k}{2^k} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(k-1)k}{2^{k-1}} + 2 \sum_{k=1}^{\infty} \frac{k}{2^k} = \frac{S_2}{2} + 2S_1$$

$$\implies S_2 = 8$$

Very likely a general path to get $S_r = \sum_{k=1}^{\infty} \frac{k(k+1)..(k+r-1)}{2^k}$

With SageMath : $S_r(r : 0..6) = 1, 2, 8, 48, 384, 3840, 46080$

and then OEIS A000165 $S_r = (2r)!!$ (not proved yet but it seems that $S_r = 2rS_{r-1}$)