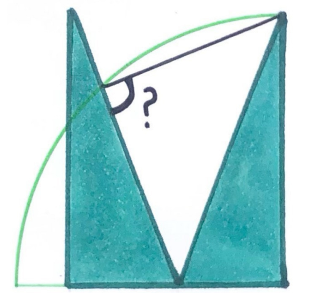
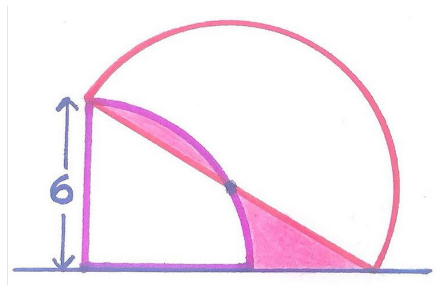
Geometric Puzzle Meditations

13 January 2024

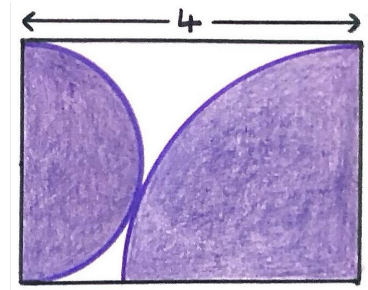
Jim Stevenson



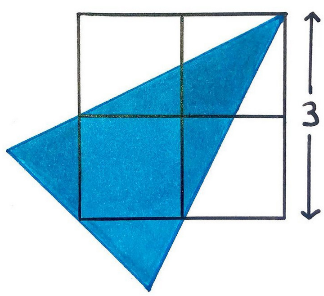
#1 The green triangles are congruent. What's the angle?



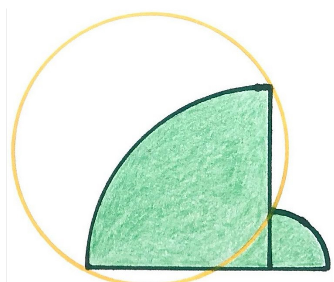
#2 The centre of the semicircle lies on the perimeter of the quarter circle. What's the shaded area?



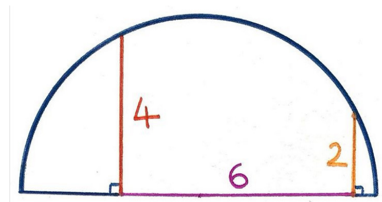
#3 What's the shaded area?



#4 Four squares and an isosceles triangle. What's the blue area?



#5 The yellow circle has radius 4. What's the total area of the two quarter circles?



#6 What's the area of this semicircle?

Since Twitter (now X) is no longer public, I was afraid I would not have access to new Catriona Agg puzzles, but she has put them up on Instagram, which is partially available to the public. I managed to find a half dozen interesting new brain ticklers.

Solution to #1[1]

First, notice that the exact location of the unknown angle along the circle is arbitrary, and so the angle must be constant.

It turns out there is a slick, one-step surprising proof that the angle must be a right angle (Figure 1). Extend the quarter circle into the entire circle and flip the right-hand triangle over the horizontal diameter. Then it is also congruent and the corresponding angles sum to 180° as shown, proving the line from the unknown angle vertex to the end of the circle diameter is a straight line. Therefore the inscribed angle in the circle must be ½ its central angle, or 90°.

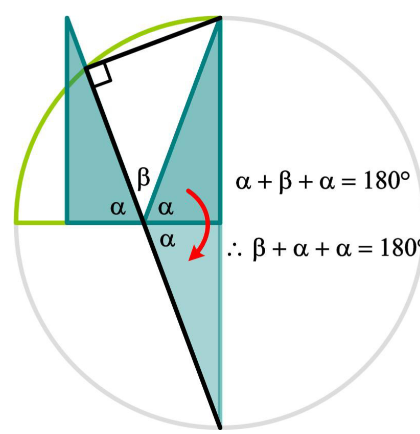


Figure 1

[1] August 12, 2023 (https://www.instagram.com/cshearer41/p/Cv1vuNQNhYV/)

Solution to #2[2]

Extend the (red) semicircle to an entire circle. Then because the angle in the triangle is 90°, it must subtend the diameter of the red circle and so lie on the circle itself. Therefore the radius of the (lavender) quarter circle is the same as the red semicircle, namely 6 (since the quarter circle passes through the center of the red circle). The dashed radius defines an equilateral triangle with the vertical edge of the right triangle and a radius of the red semicircle. So its interior angles are 60°. Therefore the complement of these angles is 30°. These values imply the horizontal leg of the right triangle is $6\sqrt{3}$.

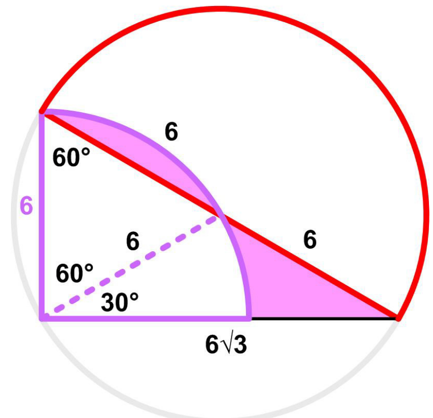


Figure 2

From all these values we conclude

S_1 = area of 60° sector	$= 6^2\pi/6$	$= 6\pi$.
S_2 = area of 30° sector	$= 6^2\pi/12$	$= 3\pi$.
T_1 = area of 60° equilateral triangle	$= 3\,(3\sqrt{3})$	$= 9\sqrt{3}$.
T_2 = area of 30° isosceles triangle	$= 3\,(3\sqrt{3})$	$= 9\sqrt{3}$.

Therefore, the desired area is

$$(S_1 - T_1) + (T_2 - S_2) = 6\pi - 9\sqrt{3} + 9\sqrt{3} - 3\pi = 3\pi$$

Solution to #3[3]

Let R be the radius of the large quarter circle and r the radius of the small semicircle (Figure 3). Then R = 2r and

$$4^2 + r^2 = (r + R)^2 = 9r^2 \Rightarrow r^2 = 2$$

So the area of the shaded region is

$$\tfrac{1}{2}\pi r^2 + \tfrac{1}{4}\pi R^2 = \pi\,{}^3/_2\, r^2 = 3\pi$$

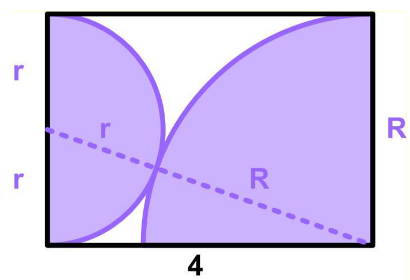


Figure 3

Solution to #4[4]

We consider the altitudes and bases of the original blue triangle and a sub-triangle as shown in Figure 4. The altitude of the original triangle is $3\sqrt{2}$ and the sub-triangle is ¾ of that or $({}^3/_2)^2\sqrt{2}$. The base of the sub-triangle is ${}^3/_2\sqrt{2}$ and the original triangle an unknown x.

Therefore we have the following proportion of altitudes to bases:

$$\frac{(\frac{3}{2})^2\sqrt{2}}{\frac{3}{2}\sqrt{2}} = \frac{3\sqrt{2}}{x}.$$

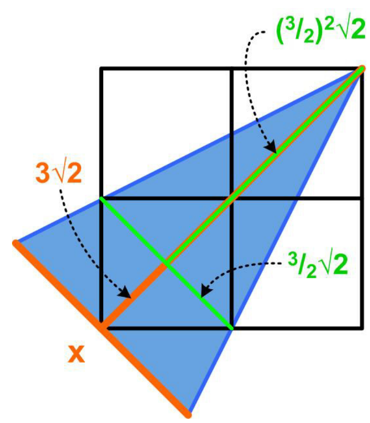


Figure 4

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[2] August 17, 2023 (https://www.instagram.com/cshearer41/p/CwCUWVTNHWZ/)
[3] February 16, 2023 (https://www.instagram.com/cshearer41/p/CouWLtqtsoq/)
[4] August 13, 2023 (https://www.instagram.com/cshearer41/p/Cv5K8Huts9x/)

So $x = 2\sqrt{2}$.

Therefore the area of the original blue triangle is

$$\tfrac{1}{2}(2\sqrt{2})(3\sqrt{2}) = 6.$$

Solution to #5[5]

Let the radius of the larger quarter circle be R and that of the smaller r. Since the sizes of the quarter sectors were not specified, we could shrink the smaller until it vanishes, leaving the larger quarter sector inscribed in the full circle (Figure 5). That would imply that the hypotenuse of the isosceles right triangle determined by the radii of the quarter sector is the diameter of the full circle, namely 2×4 = 8. So the sides of the quarter sector would be $4\sqrt{2}$, making the area

$$(4\sqrt{2})^2\,\pi/4 = 8\pi.$$

Another possible extreme is when the two sectors are the same (Figure 6). Then the radii are the same and equal the radius of the (yellow) circle 4. So the sum of areas is the area of the semicircle of radius 4, or $\pi 4^2/2 = 8\pi$ again.

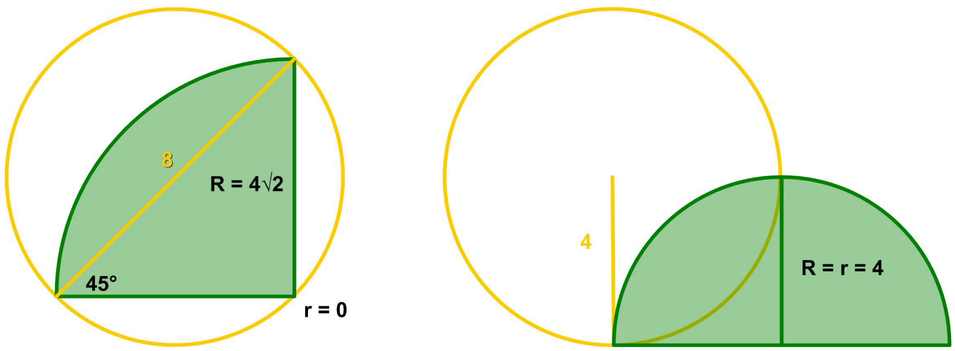

Figure 5 Figure 6

So we have our answer. However, we should prove that the sum of the quarter sectors remains constant as we change their sizes.

Notice in Figure 7 where the vertical side of the large sector cuts the circle. The length of this segment of the circle is R – r. The perpendicular bisector of this segment of the circle passes through the center of the circle (purple line segments). So one half of the vertical segment is of length (R – r)/2.

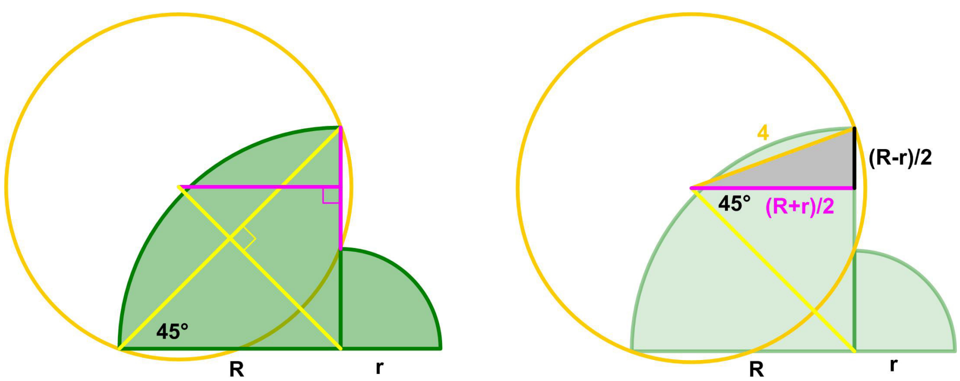

Figure 7 Figure 8

5 February 25, 2023 (https://www.instagram.com/cshearer41/p/CpFlGzStfau/)

Similarly in Figure 7 the (yellow) perpendicular bisector of the (yellow) line segment of the circle joining the ends of the large sector also passes through the center of the circle. The two bisectors, along with the vertical side of the sector, form an isosceles right triangle. This means the (purple) horizontal segment is the same length as the segment lying along the vertical side of the sector, namely r + (R – r)/2 = (R + r)/2. See Figure 8.

So we have the following result from the Pythagorean Theorem using the radius 4 of the circle:

$$4^2 = \left(\frac{R+r}{2}\right)^2 + \left(\frac{R-r}{2}\right)^2 = \frac{R^2+r^2}{2}$$

And the sum of the areas of the two quarter sectors is

$$\boxed{\frac{\pi R^2}{4} + \frac{\pi r^2}{4} = 8\pi}$$

So we have confirmed that the sum of the areas is constant for different sized sectors.

Solution to #6[6]

Let r be the radius of the semicircle and x the distance between the center of the semicircle and the foot of the vertical line of length 4 (Figure 9). Then we get two equations from the Pythagorean Theorem

$$r^2 = 2^2 + (6 - x)^2 = 40 - 12x + x^2$$

$$r^2 = 4^2 + x^2 = 16 + x^2$$

Subtracting the second from the first yields 12x = 24, so that x = 2 and $r^2 = 20$. Therefore the area of the semicircle is

$$\pi r^2 / 2 = 10\pi$$

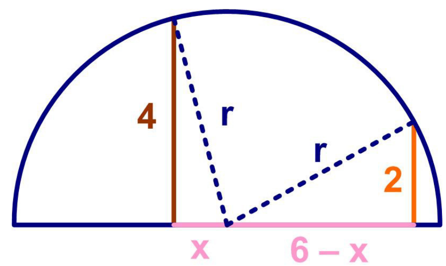

Figure 9

[6] February 16, 2023 (https://www.instagram.com/cshearer41/p/CouV_YvtyWi/)