Parallel Lines Problem

13 March 2024

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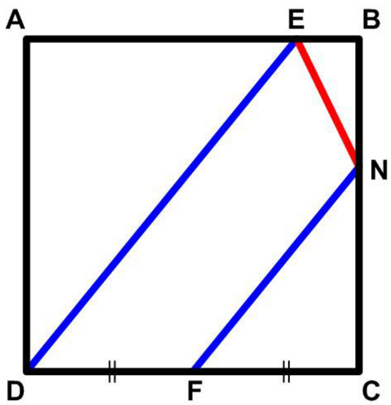

This is an interesting problem from the collection *Five Hundred Mathematical Challenges* ([1]).

Problem 251. Let $ABCD$ be a square, F be the midpoint of DC, and E be any point on AB such that $AE > EB$. Determine N on BC such that $DE \parallel FN$. Prove that EN is tangent to the inscribed circle of the square.

My Solution

First, I considered the perpendicular to the putative tangent line through the center of the square (circle) (Figure 1). The idea is to show it is of length r, the radius of the inscribed circle (= ½ the side of the square). Then that would show the line *is* tangent to the circle.

Plane Geometry Approach. Adding the horizontal (dashed) line and diagonal shown in Figure 2, it is "clear" from the diagram that the two right triangles are congruent, and so the perpendicular is of length r. But try as I might, I could not *prove* the triangles were congruent. There must be a way, but I could not see it. In particular, I could not see how to involve the two parallel blue lines.

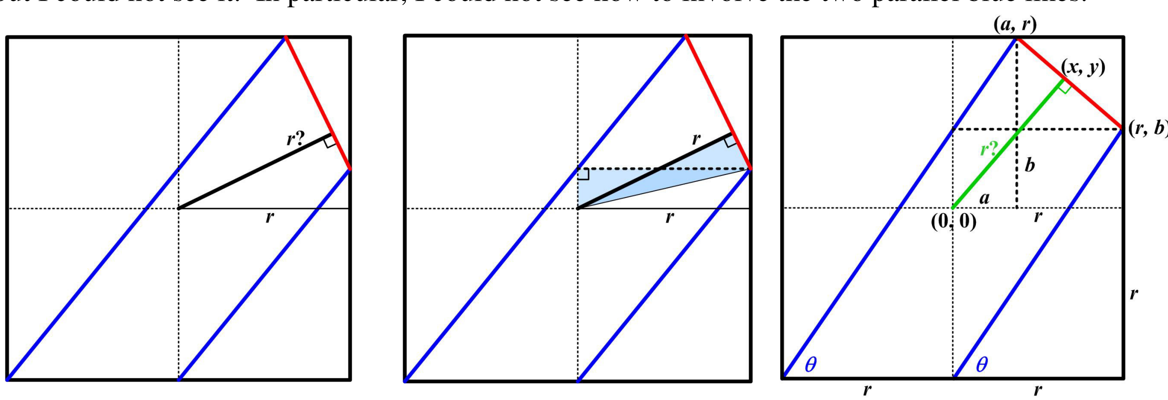

Figure 1 **Figure 2** **Figure 3**

Analytic Geometry Approach. Therefore, I resorted to a sledgehammer approach using analytic geometry, which admittedly is interesting in its own right. Consider the problem as labeled in Figure 3. Then from the parallel blue lines we have

$$\tan\theta = \frac{r+b}{r} = \frac{2r}{r+a} \;\Rightarrow\; 2r^2 = (r+a)(r+b) = r^2 + (a+b)r + ab$$

So we have
$$2r^2 - 2(a+b)r = 2ab \tag{1}$$

and
$$r^2 - ab = (a+b)r \tag{2}$$

which we will need later. (So the analytic geometry approach provided a way to involve the parallel blue lines.)

Now we develop the equations for the red and green straight lines and find their intersection point

(x, y). Let m be the slope of the red line. Then

$$m = \frac{b-r}{r-a}$$

and the red line equation is $y = m(x-a)+r$.

So the perpendicular line equation is $y = -\frac{1}{m}x$,

and so the intersection is when $-\frac{1}{m}x = m(x-a)+r$

or $$ma - r = \left(m + \frac{1}{m}\right)x = \frac{m^2+1}{m}x .$$

So $$x = \frac{m(ma-r)}{m^2+1}$$

and $$y = -\frac{ma-r}{m^2+1}$$

Now we want to show $x^2 + y^2 = r^2$.

$$x^2 + y^2 = \left(\frac{ma-r}{m^2+1}\right)^2 (m^2+1) = \frac{(ma-r)^2}{m^2+1} \tag{3}$$

From equation (2)

$$ma - r = \left(\frac{b-r}{r-q}\right)a - r = \frac{ab-r^2}{r-a} = \frac{-(a+b)r}{(r-a)}$$

and from equation (1)

$$m^2+1 = \left(\frac{b-r}{r-a}\right)^2 + 1 = \frac{2r^2 - 2(a+b)r + a^2 + b^2}{(r-a)^2} = \frac{(a+b)^2}{(r-a)^2}$$

So equation (3) becomes

$$x^2 + y^2 = \frac{(ma-r)^2}{m^2+1} = \frac{(-(a+b)r)^2}{(r-a)^2}\frac{(r-a)^2}{(a+b)^2} = r^2 ,$$

which is what we wanted to show (Figure 4).

Analytic geometry solutions are always a bit unsatisfactory, since they mask geometric reasoning behind algebraic manipulations which follow their own logic.

Apparently *Five Hundred Mathematical Challenges* couldn't find a slick plane geometry solution either. All their solutions involve a fair number of computations too.

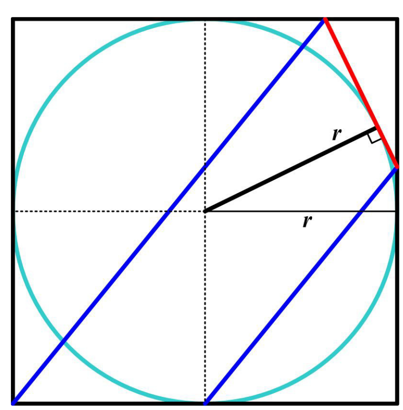

Figure 4

Problem 251. *First solution.* Our first solution is motivated by the theorem that the tangents drawn from an external point to a circle are of equal length. In Figure 162, $\overline{UP} = \overline{UR}$ and

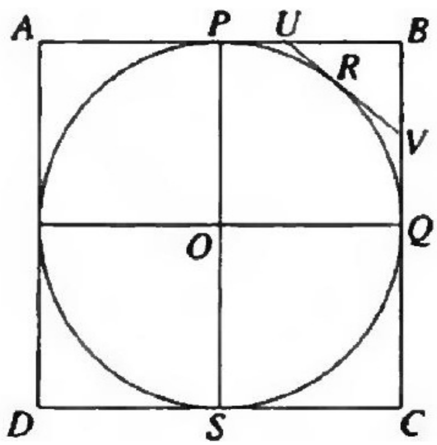

FIGURE 162

$\overline{VR} = \overline{VQ}$. Thus, $\overline{UV} = \overline{PU} + \overline{VQ}$. We now establish the converse theorem.

Proposition. *Let $ABCD$ be a square and let P and Q be the midpoints of AB and BC respectively. Suppose U and V are points in the segments PB and BQ such that $\overline{UV} = \overline{PU} + \overline{VQ}$. Then UV is tangent to the inscribed circle of the square.*

Proof. The three possibilities for the chord UV are:

(1) not to intersect the circle;

(2) to be tangent to the circle;

(3) to intersect the circle in two points.

We rule out cases (1) and (3). If either of these two cases be valid, consider a parallel chord $U'V'$ which is tangent to the circle. Then, as pointed out above, $\overline{U'V'} = \overline{PU'} + \overline{V'Q}$. In the case of (1), $\overline{UV} < \overline{U'V'}$, $\overline{PU} > \overline{PU'}$, $\overline{QV} > \overline{QV'}$, which contradicts the hypothesis. Similarly, in the case of (3), $\overline{UV} > \overline{U'V'}$, $\overline{PU} < \overline{PU'}$, $\overline{QV} < \overline{QV'}$, which again contradicts the hypothesis. Hence case (2) must be correct.

(A direct proof can be obtained by choosing R on UV so that $\overline{PU} = \overline{UR}$, $\overline{RV} = \overline{VQ}$, by then showing that $\angle BUV = 2\angle URP$, $\angle BVU = 2\angle VRQ$, and hence that $\angle PRQ = 135^\circ$ and $PQRS$ is a concyclic quadrilateral, S being the midpoint of CD.)

Returning to the problem, we let $\overline{DF} = a$, $\overline{AE} = a + p$, $\overline{CH} = a + q$ and $\overline{EH} = r$ (see Figure 163). Since $\triangle EAD$ is similar to $\triangle FCH$, $\dfrac{\overline{AD}}{\overline{AE}} = \dfrac{\overline{CH}}{\overline{CF}}$ so that

$$\frac{2a}{a+p} = \frac{q+a}{a},$$

whence

$$q = \frac{a^2 - ap}{a+p}.$$

By Pythagoras' Theorem applied to $\triangle BEH$,

$$\begin{aligned} r^2 &= (a-p)^2 + (a-q)^2 \\ &= (a-p)^2 + \left(\frac{2ap}{a+p}\right)^2 \\ &= \frac{(a^2-p^2)^2 + 4a^2p^2}{(a+p)^2} = \frac{(a^2+p^2)^2}{(a+p)^2}, \end{aligned}$$

whence

$$r = \frac{a^2+p^2}{a+p} = p + \frac{a^2-ap}{a+p} = p + q.$$

Now apply the proposition.

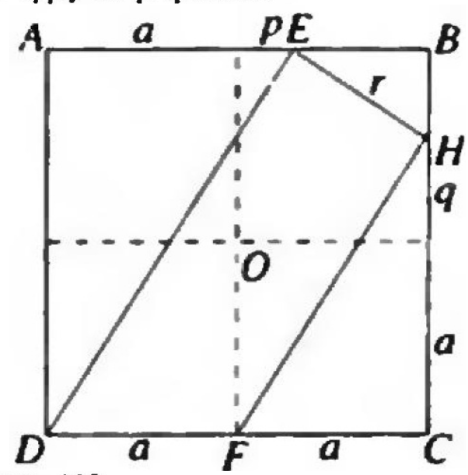

FIGURE 163

Second solution. The inscribed circle will be tangent to EH if and only if the distance from O (the center of the square) to EH is equal to a, the radius of the circle. This can be found by dividing twice the area of triangle OEH by the length of EH to get the altitude of the triangle from O. Introduce Cartesian coordinates with the origin at O and the axes parallel to the sides of the square. Then $E = (p, a)$, $H = (a, q)$ and the area of $\triangle OEH$ is $\dfrac{a^2 - pq}{2}$.

As in the first solution, the length of EH is $p + q$, and $pq = a^2 - a(p + q)$. Thus $a^2 - pq = a(p + q)$. Hence the distance from O to EH is $\dfrac{a^2 - pq}{p + q} = a$, as required.

Third solution. Introduce Cartesian coordinates with the origin at O as in the second solution. The line through E and H has equation

$$(a - q)x + (a - p)y = a^2 - pq,$$

and the distance from the origin to this line is

$$\frac{a^2 - pq}{\sqrt{(a - q)^2 + (a - p)^2}} = \frac{a(p + q)}{p + q} = a,$$

where the computations are as in the first solution. Thus the distance from O to EH is equal to the radius of the inscribed circle.

Fourth solution. (E. Michael Thirian) Let the length of a side of the square be 2. Let $\overline{EB} = u$ and $\overline{BH} = v$ so that $\overline{AE} = 2 - u$ and $\overline{CH} = 2 - v$. Let OK and OM be the perpendiculars from the center O of the square to EH and BC respectively. EH and OM produced meet in L. See Figure 164. Since $\triangle AED \sim \triangle CFH$, $\overline{AE} \cdot \overline{CH} = \overline{AD} \cdot \overline{FC}$, $(2 - u)(2 - v) = 2$, or $uv - 2u - 2v + 2 = 0$. From the similar triangles EBH and LKO, $\overline{EH} \cdot \overline{KO} = \overline{BH} \cdot \overline{LO}$. Since $\overline{LO} = 1 + \dfrac{u(1 - v)}{v} = \dfrac{u + v - uv}{v}$,

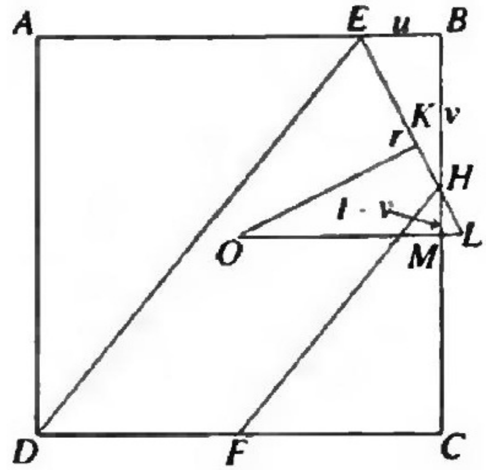

FIGURE 164

$$\overline{KO}^2 = \frac{v^2 \left\{\frac{u+v-uv}{v}\right\}^2}{u^2 + v^2} = \frac{(u + v - uv)^2}{u^2 + v^2}$$
$$= \frac{u^2 + v^2 + uv(uv - 2u - 2v + 2)}{u^2 + v^2}$$
$$= 1.$$

Thus, the distance from O to EH is equal to the distance from O to a side of the square and the result follows.

Remark. By considering an orthogonal projection of the entire figure, we also have the following equivalent result. Suppose $ABCD$ is a parallelogram circumscribing an ellipse which touches the parallelogram at the midpoints of the sides. If F is the midpoint of DC, H is on BC and E is on AB with $FH \parallel DE$, then EH is tangent to the conic.

References

[1] Barbeau, Edward J., Murray S. Klamkin, William O. J. Moser, *Five Hundred Mathematical Challenges*, Spectrum Series, Mathematical Association of America, Washington D.C, 1995