# Making Arrows

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This is an interesting problem from 180 BC China ([1]).

In one day, a person can make 30 arrows or fletch [put the feathers on] 20 arrows. How many arrows can this person both make and fletch in a day?

It turns out the solution to this problem led me into the history of numerator/denominator (aka common) fractions, a subject I had been finding difficult to track down.

## Solution

Let $r_A = 30$ be the rate of making arrows per day, $r_F = 20$ the rate of fletching arrows per day, $t_A$ the fraction of day spent making arrows, and N the number of arrows made and fletched in one day. Then

$$N = r_A\, t_A = r_F\,(1 - t_A)$$

since the arrow maker is not making fletches when he is making an arrow. Solving this equation for $t_A$ yields $t_A = r_F /(r_A + r_F)$ so

$$N = r_A\, r_F /(r_A + r_F) = 1 / (1/r_A + 1/r_F) = 1 / (5/60) = 12$$

**Comment**. It was not clear to me how the Chinese solved this problem before the advent of symbolic algebra that we used. Nor did I think that they knew how to manipulate numerator/denominator (common) type fractions. So imagine my surprise when I discovered the Chinese did know about common fractions at least by the second century BC. We shall see this is also before the Indian discussion of them in Brahmagupta's 628 AD treatise *Brāhma-spuṭha-siddhānta* and the Bakhshālī Manuscript.

## Chinese Fractions

Cullen writes ([2] p.1):

> The *Suàn shù shū* 筭數書 is an ancient Chinese collection of writings on mathematics approximately seven thousand characters in length, written on 190 bamboo strips. It was discovered together with other writings in 1983 when archaeologists opened a tomb at Zhāngjiāshān in Húběi province. From documentary evidence this tomb is thought to have been closed in 186 BC, early in the Western Hàn dynasty. …
>
> … The *Suàn shù shū* itself is certainly the oldest Chinese excavated text with substantial mathematical content. …
>
> … the earliest Chinese mathematical text known to us before the discovery of the *Suàn shù shū* … [is] the *Jiǔ zhāng suàn shù* 九章算術, 'Mathematical procedures under nine headings' or 'Nine chapters on mathematical procedures'—commonly known amongst Western scholars as the 'Nine Chapters'. This work is usually thought to have reached its final form around the first century AD, and has a number of parallels of content with the *Suàn shù shū*.

Details of the Chinese manipulation of common fractions is given in the Appendix below (p.5).

# Indian Fractions

From Datta and Singh's classic 1938 work on Indian mathematics ([3] p.185):

> **Early Use.** In India, the knowledge of fractions can be traced back to very early times. In the oldest known work, the *Rigveda*,[1] the fractions one-half (*ardha*) and three-fourths (*tri-pada*[2]) occur. In a passage of the *Maitrayani Samhita*[3] are mentioned the fractions one-sixteenth (*kala*), one-twelfth (*kustha*), one-eighth (*sapha*) and one-fourth (*pada*). In the earliest known mathematical works, the *Sulba-sutra*,[4] fractions have not only been mentioned, but have been used in the statement and solution of problems.[5]
>
> The ancient Egyptians and Babylonians are known to have used fractions with unit numerators, but there is little evidence of the use by these people of what are called composite fractions. …

Furthermore ([3] p.188-9),

> **Writing of Fractions.** From very early times (c.200 A.D.) the Hindus wrote fractions just as we do now, but without the dividing line. When several fractions occurred in the same problem, they were in general separated from each other by vertical and horizontal lines.

Sykorova ([4]) provides a description of Indian fractions and their arithmetic operations. The discussion makes the following statement about sources ([4] p.133):

> **Historical sources**
>
> The best known mathematical texts containing fractions are as follows. Fractions were used in Bakhshālī manuscript (circa 400 AD)[6]—the anonymous mathematical work written on birch-bark. The rules for arithmetic with fractions were described especially by Brahmagupta (circa 598–670) in his work *Brāhma-spuṭha-siddhānta*, Mahāvīra (circa 800–870) in his work *Gaṇita-sāra-saṁgraha*, Śrīdhara (circa 870–930) in his work *Triśatikā*, Śrīpati (1019–1066) in his work Ganita-tilaka and Bhāskara II (1114–1185) in his book *Līlāvatī*.

These sources describe the numerical operations on fractions we are familiar with today. It is not clear to me if earlier sources, such as the *Śulba-sūtras*, do this, even though they mention fractions. So the earliest written example seems to be the Bakhshālī manuscript, if the 400 date is valid, otherwise it would be Brahmagupta's 628 *Brāhma-spuṭha-siddhānta*. Indications are that these numerical operations were known earlier, but there doesn't seem to be written documentation showing that.

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1 JOS: ("Rigveda" *Wikipedia*) "The core text, known as the *Rigveda Samhita*, is a collection of 1,028 hymns (sūktas) in about 10,600 verses (called ṛc, eponymous of the name Rigveda), organized into ten books (maṇḍalas). … Philological and linguistic evidence indicates that the bulk of the *Rigveda Samhita* was composed in the northwestern region (Punjab) of the Indian subcontinent, most likely between c.1500 and 1200 BC, although a wider approximation of c.1700–1100 BC has also been given." (https://en.wikipedia.org/wiki/Rigveda, retrieved 8/31/2019)

2 (Original footnote) *RV*, x. 90. 4.

3 (Original footnote) iii. 7. 7. JOS: (*Wikipedia*) "The Maitrayani saṃhitā is the oldest Yajurveda Samhita that has survived, … The core text of the Yajurveda falls within the classical Mantra period of Vedic Sanskrit at the end of the 2nd millennium BCE … The scholarly consensus dates the bulk of the Yajurveda and Atharvaveda hymns to the early Indian Iron Age, c. 1200 or 1000 BC …"

4 JOS: (*Wikipedia* 8/26/2019) "The *Śulba Sūtras* … (c.700–400 BCE) list rules for the construction of sacrificial fire altars. … They are the only sources of knowledge of Indian mathematics from the Vedic period."

5 (Original footnote) B. Datta, *Sulba*, pp. 212 ff.

6 JOS: This date is challenged. See the discussion below p.3.

## Bakhshālī Manuscript

Concerning the c.400AD date for the manuscript, Robertson in 2000 says ([5]):

> I [EFR] feel that if one weighs all the evidence of these experts the most likely conclusion is that the manuscript is a later copy of a work first composed around 400 AD. Why do I believe that the actual manuscript was written later? Well our current understanding of Indian numerals and writing would date the numerals used in the manuscript as not having appeared before the ninth or tenth century. To accept that this style of numeral existed in 400 AD. would force us to change greatly our whole concept of the time-scale for the development of Indian numerals. Sometimes, of course, we are forced into major rethinks but, without supporting evidence, everything points to the manuscript being a tenth century copy of an original from around 400 AD. Despite the claims of Kaye, it is essentially certain that the manuscript is Indian.
>
> The attraction of the date of 400 AD for the Bakhshali manuscript is that this puts it just before the "classical period" of Indian mathematics which began with the work of Aryabhata I around 500. It would then fill in knowledge we have of Indian mathematics for, prior to the discovery of this manuscript, we had little knowledge of Indian mathematics between the dates of about 200 BC. and 500 AD. This date would make it a document near the end of the period of Jaina mathematics[7] and it can be seen as, in some sense, marking the achievements of the Jains.

But in 2017 the Bodleian Library performed carbon-dating tests on the birch-bark that the manuscript was written on (but not on the ink that was used). As a result, they obtained three different centuries and empires: AD 224–383 (Indo-Scythian), 680–779 (Turk Shahis), and 885–993 (Saffarid dynasty). From Wikipedia ([6]):

> Prior to the proposed radiocarbon dates of the 2017 study, most scholars agreed that the physical manuscript was a copy of a more ancient text, whose date had to be estimated partly on the basis of its content. Hoernlé thought that the manuscript was from the 9th century, but the original was from the 3rd or 4th century.[8] Indian scholars assigned it an earlier date. Datta assigned it to the "early centuries of the Christian era".[9] Channabasappa dated it to AD 200–400, on the grounds that it uses mathematical terminology different from that of Aryabhata.[10] Hayashi noted some similarities between the manuscript and Bhaskara I's work (AD 629), and said that it was "not much later than Bhaskara I".[11]

Kim Plofker et al. ([7]) challenged the conclusions from the Bodleian Library. They estimated the date for the MS to be in the second half of the first millennium, that is, after 500 AD and summarized their views as follows:

- The proposed division of the Bakhshālī Manuscript text into three chronologically distinct sections corresponding to the three radiocarbon date ranges is contradicted by the unified appearance of its content and writing. If its birch-bark leaves do indeed differ widely in age, the date of the youngest folio is logically the (approximate) date of the scribal activity. This fits well with past estimates of the date of the Bakhshālī Manuscript based on historical, philological and palaeographic arguments.

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[7] http://www-history.mcs.st-andrews.ac.uk/HistTopics/Jaina_mathematics.html

[8] (original footnote) G. R. Kaye, on the other hand, thought in 1927 that the work was composed in the 12th century, but this was discounted in recent scholarship. G. G. Joseph wrote, "It is particularly unfortunate that Kaye is still quoted as an authority on Indian mathematics."

[9] (original footnote) Bibhutibhusan Datta (1929). "Book Review: G. R. Kaye, The Bakhshâlî Manuscript—A Study in Mediaeval Mathematics, 1927". *Bull. Amer. Math. Soc*. 35 (4): 579–580

[10] (original footnote) "London museum showcases India's contribution to science". www.thehindubusinessline.com. Retrieved 3 February 2022.

[11] (original footnote) Takao Hayashi (2008), "Bakhshālī Manuscript", in Helaine Selin (ed.), *Encyclopaedia of the History of Science, Technology, and Medicine in Non-Western Cultures*, vol. 1, Springer, pp. B1–B3

- Bakhshālī Manuscript, considered as the carrier of a unified text, includes a concept of written zeros that function as arithmetical operators, i.e., as numbers in their own right, and not merely as place-holder digits. This too fits well with the manuscript's generally-accepted dating to the second half of the first millennium CE.
- Attempting to trace the historical development of mathematical concepts such as the zero and decimal place value solely or primarily through ancient physical evidence is a fundamentally unreliable enterprise. The historical significance of the Bakhshālī Manuscript and its mathematical content cannot be understood by isolated speculative inferences based on the apparent physical age(s) of the bark it was written on: it requires careful comparison with related ideas in a long sequence of other Indic texts treating various concepts associated with calculation (gaṇita).

# Conclusion

This labored discussion of dates was my attempt at suggesting that perhaps the Indian ideas about operating with fractions may have benefited from earlier Chinese ideas, as Plofker suggested regarding negative numbers.

And according to Cullen the Chinese also used a base 10 number system, though without numerals distinct from words to represent the numbers ([2] pp.24-5):

> When however numbers were written down in Chinese as part of normal prose, as in the *Suàn shù shū*, they were represented using a basic set of characters for the numbers 1 to 9, but without the use of a common marker for zero, so that simple place-value cannot be applied. Instead multiples of ten are specified in the number, so that the number 57,982 is written *wǔ wàn qī qiān jiǔ bǎi bā shí èr* literally 'five myriads, seven thousands, nine hundreds, eight tens, two' and 6003 would be *liù qiàn sān* 'six thousands, three'. In pre-modern Chinese the distinction between words for numbers and figures to represent them therefore does not exist in normal writing.

In any case, the West owes a debt of gratitude to the East for its decimal number system, zero, negative numbers, and manipulation of common fractions. Not everything came from the Greeks.

# References


[1] "Making Arrows," *Convergence*, Mathematical Association of America, March 2007. From *Suan shu shu* (*Writings on Reckoning*), p.79. c. 180 BCE ([2])
(New Link: https://old.maa.org/press/periodicals/convergence/making-arrows, retrieved 7/2/2023).

[2] Cullen, Christopher, *The* Suàn shù shū 筭數書*, 'Writings on reckoning': A translation of a Chinese mathematical collection of the second century BC, with explanatory commentary*, Needham Research Institute Working Papers: 1, Needham Research Institute, Cambridge, UK, 2004. (https://www.nri.org.uk/suanshushu.html, retrieved 7/2/2023)

[3] Datta, Bibhutibhusanl and Avadhesh Narayan Singh, *History of Hindu Mathematics, A Source Book Parts I and II* (Numeral notation and arithmetic), Asia Publishing House, 1935, 1938, 1962. pp.185-203 (https://archive.org/details/wg143, retrieved 7/13/2021)

[4] Sykorova, I. , "Fractions in Ancient Indian Mathematics", *WDS'10 Proceedings of Contributed Papers, Part I,* 2010. pp.133–138.
(https://www.mff.cuni.cz/veda/konference/wds/proc/pdf10/WDS10_122_m8_Sykorova.pdf, retrieved 8/4/2020)

[5] O'Connor, J. J., and E. F. Robertson, "The Bakhshali manuscript", *MacTutor History of Mathematics*, November 2000. (http://www-history.mcs.st-andrews.ac.uk/HistTopics/Bakhshali_manuscript.html, retrieved 7/16/2019)

[6] "Bakhshali Manuscript", *Wikipedia*. (https://en.wikipedia.org/wiki/Bakhshali_manuscript, retrieved 6/16/2024)

[7] Plofker, Kim, Agathe Keller, Takao Hayashi, Clemency Montelle and Dominik Wujastyk, "The Bakhshālī Manuscript: A Response to the Bodleian Library's Radiocarbon Dating," *History of Science in South Asia*, 5.1, (2017) pp.134–150. (https://journals.library.ualberta.ca/hssa/index.php/hssa/article/view/22, retrieved 6/17/2024)

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# Appendix: Chinese Operations on Fractions in *Writings on Reckoning* c.186 BC

It turns out that all the "modern" operations on fractions are presented in the Chinese *Writings on Reckonings*, though in verbal from rather than symbolic form ([2]). For fractions Cullen translates the original Chinese term "mother" as "denominator" and "child" as "numerator".

| Operation | Symbolic | Verbal |
|---|---|---|
| Multiplication: | a/b × c/d = ac/bd | p.36, S7 (2) **Parts multiplying**<br><br>The method for a part multiplying a part [is] always: The denominators multiply together to make the divisor; the numerators multiply together to make the dividend. |
| Simplifying: | 14/35 = 2·7/5·7 = (2/5)(7/7) = 2/5 | p.38, S17, S18 (7) **Simplifying parts**<br><br>(c) Take the numerator of the part from the denominator. [If that is] the lesser take the denominator from the numerator. When [the numbers on the sides of] the numerator and denominator are equal, take that [number] as the divisor. For numerator and denominator complete one for [each time] they accommodate the divisor.<br><br>(The point of the alternating subtractions in the present case is of course to find a number that is a factor of both the numerator and denominator.)<br><br>[JOS: 35 – 14 = 21, 21 – 14 = 7, 14 – 7 = 7 ⇒ 7 is factor. (5·7 – 2·7 = 3·7, 3·7 – 2·7 = 7, 2·7 – 1·7 = 7)] |
| Addition: | a/b + c/d = (ad + bc)/bd | p.40, S21, S22, S23, S24, S25 (8) **Joining parts**<br><br>(b) In a case where denominators are of the same kind as one another, numerators go with one another [in addition]; [For] those not of the same kind as each other, multiply the denominators together to make the divisor. The numerators multiply the opposite denominators and combine to make the dividend. Complete one for [each time the dividend] accommodates the divisor. |

| Operation | Symbolic | Verbal |
|---|---|---|
| Addition: | $7\frac{1}{3} + \frac{1}{2} = {}^{22}/_{3} + {}^{1}/_{2} = {}^{(44+3)}/_{6} = {}^{47}/_{6}$ | p.40, S21, S22, S23, S24, S25 (8) **Joining parts [Mixed Fractions]** |
| | ${}^{47}/_{6} / 5 = {}^{47}/_{30} = 1\,{}^{17}/_{30}$ | (d) Five men divide 7 cash and a diminished half [= ${}^{1}/_{3}$], and a half cash. A man gets 1 cash and 17/30 cash. The method: in the lowest [place there is] a third, [so] make 6 from 1; then go on to six-fold the [number of] men to make the divisor; likewise six-fold the [number of] cash[12] to make the dividend. |
| Subtraction: | a/b – c/d = (ad – bc)/bd | p.44, S28, S29 (10) **Paying out gold** |
| | | (a) … The method: The denominators are multiplied together to make the divisor; the numerators multiply the denominators reciprocally, each making a dividend of its own; diminish it by the [amount] paid out; then the remainder is the remaining [gold]. |
| Division: | (a/b) / (c/d) = ad/bc | p.106, S162, S163 [(66) **Revealing the length**] |
| | | (d) The method for seeking the length: the numerator of the breadth part multiplies the denominator of the area denominator to make the divisor; The area part numerator multiplies the breadth part denominator to make the dividend; … |



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[12] JOS: I admit I didn't understand this statement. The dividend (numerator) as I have shown computed is more complicated than just multiplying by 6, unless the meaning is $6\cdot(7 + {}^{1}/_{3} + {}^{1}/_{2}) = (42 + 2 + 3) = 47$.