Horses to Qi

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Jim Stevenson

This is a fairly challenging problem from the c.100AD Chinese mathematical work, *Jiŭ zhāng suàn shù* (*The Nine Chapters on the Mathematical Art*) found at the MAA *Convergence* website ([1]).

Now a good horse and an inferior horse set out from Chang'an to Qi. Qi is 3000 *li* from Chang'an. The good horse travels 193 *li* on the first day and daily increases by 13 *li*; the inferior horse travels 97 *li* on the first day and daily decreases by ½ *li*. The good horse reaches Qi first and turns back to meet the inferior horse. Tell: how many days until they meet and how far has each traveled?

The solution involves common fractions, which the Chinese were already adept at using.[1]

Solution

Figure 1 shows the space-time diagram for the traveling horses. Let S_t be the distance the slow horse traveled in time t and F_t the distance the fast horse traveled in time t. Let T be the time they meet. Then T is made up of n integral days and x fraction of a day.

Figure 1

Now we have

$$S_T + (F_T - 3000) = 3000$$

or

$$S_T + F_T = 6000 \qquad (1)$$

Whole Day Computations

Let's consider first the distances traveled in whole days n. Then, for example, after 4 days the slow horse will have traveled

$$S_4 = 97 + (97 - 1/2) + (97 - 1/2 - 1/2) + (97 - 1/2 - 1/2 - 1/2) = 4 \cdot 97 - ½\,(1 + 2 + 3)$$

or after n days the slow horse will have traveled

$$S_n = 97n - ½\,(1 + 2 + 3 + \ldots + (n - 1)) = 97n - ½\,(n\,(n - 1)/2) \qquad (2)$$

Similarly, for the fast horse

$$F_n = 193n + 13\,(1 + 2 + 3 + \ldots + (n - 1)) = 193n + 13\,(n\,(n - 1)/2) \qquad (3)$$

Therefore, from equation (1) we are looking for the largest whole number n such that

$$S_n + F_n \le 6000 \text{ and } S_{n+1} + F_{n+1} > 6000$$

That is,

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[1] See my post on "Making Arrows" (https://josmfs.net/2024/06/22/making-arrows/) where I delve into the history of common fractions in Chinese mathematics, as well as a brief mention of Indian fractions.

$$S_n + F_n = 290n + 25/4\,(n^2 - n) = 25/4\,n^2 + 1135/4\,n \le 6000$$

Let's consider this expression depending on a continuous time t in order to find the value t such that we have equality, that is,

$$S_t + F_t = 25/4\,t^2 + 1135/4\,t = 6000 \tag{4}$$

or

$$25\,t^2 + 1135\,t - 24000 = 0$$

or

$$5\,t^2 + 227\,t - 4800 = 0$$

Then

$$t = \frac{-227 \pm \sqrt{227^2 + 20\cdot 4800}}{10} = \frac{-227 + 384.0950403}{10} = 15.70950403$$

So the largest whole number is 15, which means the horsemen traveled 15 days plus a fraction of a day.

Fraction of Day Computations

So each horseman traveled for some fraction x of the 16th day when they meet. We assume each horseman is riding at a constant speed for the whole day, though it is a different speed from the previous days. Then

$$S_{16} - S_{15} = \text{(constant speed)(1 day)} \Rightarrow (S_{16} - S_{15})x = \text{(constant speed)}(x \text{ fraction of day})$$

and

$$F_{16} - F_{15} = \text{(constant speed)(1 day)} \Rightarrow (F_{16} - F_{15})x = \text{(constant speed)}(x \text{ fraction of day})$$

Thus equation (1) becomes for $T = 15 + x$,

$$S_{15} + F_{15} + x((S_{16} - S_{15}) + (F_{16} - F_{15})) = 6000 \tag{5}$$

Now from equation (2)

$$S_{15} = (97 - 14/4)\cdot 15 = 1402.5\ li$$

and from equation (3)

$$F_{15} = (193 + 13\cdot 14/2)\cdot 15 = 4260\ li$$

From equation (2) we get

$$S_{n+1} - S_n = 97 - \tfrac{1}{2}\,n \Rightarrow S_{16} - S_{15} = 97 - \tfrac{1}{2}\cdot 15 = 89.5\ li$$

and from equation (3) we get

$$F_{n+1} - F_n = 193 + 13\,n \Rightarrow F_{16} - F_{15} = 193 + 13\cdot 15 = 388\ li$$

So equation (5) becomes

$$1402.5 + 4260 + x(89.5 + 388) = 6000$$

Therefore

$$x = 337.5/477.5 = 135/191$$

And so the horsemen traveled $T = 15\ {}^{135}/_{191}$ days. The good fast horse traveled

$$F_T = F_{15} + x\,(F_{16} - F_{15}) = 4260 + ({}^{135}/_{191})388 = 4534\ {}^{46}/_{191}\ li$$

and the inferior slow horse traveled

$$S_T = S_{15} + x\,(S_{16} - S_{15}) = 1402.5 + ({}^{135}/_{191})\ 89.5 = 1465\ {}^{145}/_{191}\ li$$

Comment. Note that the answer for the time $15\ ^{135}/_{191} = 15.70681$, which is the solution from equation (5) and the answer given in the *Nine Chapters*, does not equal the time $t = 15.70950$, which is the solution to the quadratic equation (4), though they are close. Initially, I thought they would be the same.

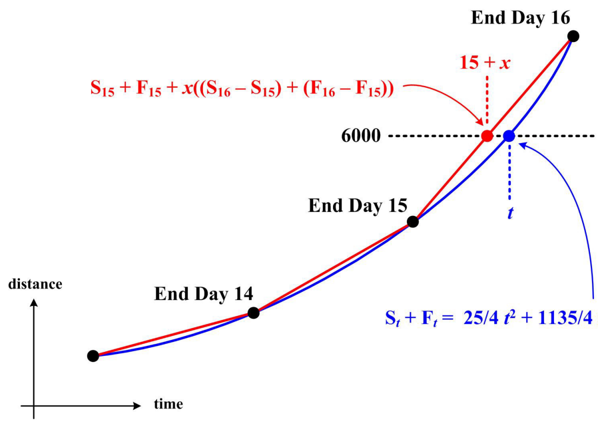

Figure 2

But a closer look at the nature of the two equations explains the difference (Figure 2). Equation (4) is a smooth parabolic curve, whereas equation (5) represents a linear interpolation along a curve made up of straight line segments. And since the parabolic curve is concave up, the time $15 + x$ reached by the line segment at 6000 *li* is less than the time t when the parabola reaches 6000 *li*. One might even argue that the parabolic shape is closer to a physical continuous change in speed of the horses than the broken line curve, but that is not how the original problem was interpreted apparently.

References

[1] "Horses to Qi," *Convergence*, Mathematical Association of America, November 2006. From *Jiǔ zhāng suàn shù* (*The Nine Chapters on the Mathematical Art*) c. 100 AD ([2] p.1)
(New link: https://old.maa.org/press/periodicals/convergence/horses-to-qi).

[2] Cullen, Christopher, *The* Suàn shù shū 筭數書*, 'Writings on reckoning': A translation of a Chinese mathematical collection of the second century BC, with explanatory commentary*, Needham Research Institute Working Papers: 1, Needham Research Institute, Cambridge, UK, 2004. (https://www.nri.org.uk/suanshushu.html)